XII. *The Oscillations of a Rotating Ellipsoidal Shell containing Fluid.*

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Introduction.

In a paper published in 'Acta Mathematica,' vol. 16, M. Folie announces the fact that the latitude of places on the earth's surface is undergoing periodic changes in a period considerably in excess of that which theory has hitherto been supposed to require. This result has been confirmed in a remarkable manner by Dr. S. C. Chandler in America (*vide* 'Astronomical Journal,' vols. 11, 12), who, as the result of an exhaustive examination of almost all the available records of latitude observations for the last half-century, has assigned 427 days as the true period in which the changes are taking place.

The old theory, based on the assumption that the earth was rigid throughout, led to a period of 305 days, and M. Folie proposes to account for the extension of this period by attributing a certain amount of freedom to the internal portions of the earth. The earth he supposes to be composed of "a solid shell moving more or less freely on a nucleus consisting of fluid at least at its surface." The argument advanced by M. Folie in favour of this constitution of the earth, namely, the independence of the motions of the shell and the nucleus, appeared to me to be unsatisfactory, and I therefore proposed to myself to test the validity of it by examining a particular case which lent itself to mathematical analysis, namely, that in which the internal surface of the shell is ellipsoidal and the nucleus consists entirely of homogeneous fluid.

The principal axes of the shell and of the cavity occupied by fluid are assumed to be coincident, and the oscillations are considered about a state of steady motion in which the axis of rotation coincides with one of these axes. It is clear that a steady motion will be possible in this case, and that such a motion will be secularly stable in the event of the axis of rotation being the axis of greatest moment for both the shell and the cavity.

The problem was originally treated by the analysis used by Poincaré in his memoir on the stability of the fluid ellipsoid with a free surface ('Acta Mathematica,'

vol. 7). This analysis reduces the determination of the motion of the fluid to the problem of finding a single function ψ, subject to certain boundary conditions, which in our case take a very simple form. In the case where the surface of the fluid is ellipsoidal, it is found that, when the system is oscillating in one of its normal modes, ψ will be expressible as the sum of a series of Lamé products of a single order n only.

When n is different from 2, the types of oscillation are such that no disturbance of the shell is involved, and a period equation for the oscillations of the fluid may be deduced in a manner similar to that given by POINCARÉ.

The types of oscillation corresponding to $n = 2$ demand exceptional treatment, in consequence of the motion communicated to the shell when they exist. The fluid motion, however, is found to be such that the molecular rotation is everywhere the same. Mr. BRYAN has suggested to me that this circumstance may be made use of in order to treat the oscillations which involve motions of the shell by a simple analysis previously employed by GREENHILL ('Proc. Camb. Phil. Soc.,' vol. 4, p. 4) which does not involve Lamé functions. To facilitate the reading of the paper, the results are first deduced by this method, and the Lamé analysis by which they were originally obtained is reserved for an appendix.

The oscillations under consideration are found to be of two types. One of these corresponds to an oscillation previously discussed by HOPKINS in his "Researches in Physical Geology" ('Phil. Trans.,' 1839). This exists only in consequence of the contained fluid, and in it the oscillations of the shell are similar in character to the "forced" nutations of the earth produced by the action of the sun and moon. In the other type the motion of the shell is closely analogous to the motion of a rigid body when slightly disturbed from a motion of pure rotation about a principal axis, and, in fact, identifies itself with such an oscillation in the event of the inertia of the fluid becoming negligible.

On applying the problem to the case of the earth, the latter mode is that on which the variations of latitude depend. The period, however, is found to be shorter than it would be if the fluid were solidified, and thus, in this particular case, M. FOLIE'S results are contradicted. It appears to me to be highly probable that any such freedom in the interior of the earth as that supposed by M. FOLIE, provided the surface does not undergo deformation, would have the effect of reducing, instead of extending, the period, and the true explanation of the phenomenon is probably that given by NEWCOMB ('Monthly Notices of the Royal Astronomical Society,' March, 1892), who shows that the elasticity of the earth, as a whole, would have the effect of prolonging the period.

§ 1. *The Period Equation.*

Let us refer to rectangular axes coincident with the principal axes of the ellipsoidal cavity.

Let α, β, γ be the principal semi-axes of this cavity; A, B, C the principal moments of inertia of the shell.

Suppose the motion of the fluid at any instant consists of a rigid body rotation with angular velocities ξ, η, ζ about the axes of the ellipsoid, compounded with the irrotational motion consequent on giving the shell additional angular velocities Ω_1, Ω_2, Ω_3.

The velocity-potential of the irrotational motion will be

$$\frac{\beta^2-\gamma^2}{\beta^2+\gamma^2}yz\Omega_1+\frac{\gamma^2-\alpha^2}{\gamma^2+\alpha^2}zx\Omega_2+\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}xy\Omega_3.$$

The velocity-components will therefore be

$$\left.\begin{aligned} u &= \frac{\gamma^2-\alpha^2}{\gamma^2+\alpha^2}z\Omega_2+\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}y\Omega_3-y\zeta+z\eta \\ v &= \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}x\Omega_3+\frac{\beta^2-\gamma^2}{\beta^2+\gamma^2}z\Omega_1-z\xi+x\zeta \\ w &= \frac{\beta^2-\gamma^2}{\beta^2+\gamma^2}y\Omega_1+\frac{\gamma^2-\alpha^2}{\gamma^2+\alpha^2}x\Omega_2-x\eta+y\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence, if h_1, h_2, h_3 be the components of angular momentum, and M denote the mass of the fluid, ρ_1 its density,

$$\left.\begin{aligned} h_1 &= \mathrm{A}(\Omega_1+\xi)+\iiint\rho_1\,dx\,dy\,dz\,(wy-vz)=\mathrm{A}(\Omega_1+\xi)+\nu\frac{\beta^2-\gamma^2}{\beta^2+\gamma^2}\Omega_1 \\ &\qquad\qquad +\tfrac{1}{5}\mathrm{M}(\beta^2+\gamma^2)\,\xi, \\ h_2 &= \mathrm{B}(\Omega_2+\eta)+\iiint\rho_1\,dx\,dy\,dz\,(uz-wx)=\mathrm{B}(\Omega_2+\eta)+\mu\frac{\alpha^2-\gamma^2}{\alpha^2+\gamma^2}\Omega_2 \\ &\qquad\qquad +\tfrac{1}{5}\mathrm{M}(\gamma^2+\alpha^2)\,\eta, \\ h_3 &= \mathrm{C}(\Omega_3+\zeta)+\iiint\rho_1\,dx\,dy\,dz\,(vx-uy)=\mathrm{C}(\Omega_3+\zeta)+\tfrac{1}{5}\mathrm{M}\frac{(\alpha^2-\beta^2)^2}{\alpha^2+\beta^2}\Omega_3 \\ &\qquad\qquad +\tfrac{1}{5}\mathrm{M}(\alpha^2+\beta^2)\,\zeta \end{aligned}\right\} \quad (2),$$

where

$$\mu=\tfrac{1}{5}\mathrm{M}(\alpha^2-\gamma^2),\quad \nu=\tfrac{1}{5}\mathrm{M}(\beta^2-\gamma^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

If the system be disturbed from a motion of pure rotation, with angular velocity ω, about the axis of z; ξ, η, Ω_1, Ω_2, Ω_3 will all be small quantities, while ζ will be approximately equal to ω, and hence, on omitting small quantities of the second order and putting $\zeta=\omega$ in small terms, the equations of angular momentum, viz. :—

$$\left.\begin{aligned} \dot{h}_1-h_2r+h_3q &= 0 \\ \dot{h}_2-h_3p+h_1r &= 0 \\ \dot{h}_3-h_1q+h_2r &= 0 \end{aligned}\right\} \text{ where } \left\{\begin{aligned} p &= \Omega_1+\xi \\ q &= \Omega_2+\eta \\ r &= \Omega_3+\zeta \end{aligned}\right.$$

become

$$A(\dot{\Omega}_1 + \dot{\xi}) + \nu \frac{\beta^2 - \gamma^2}{\beta^2 + \gamma^2} \dot{\Omega}_1 + \tfrac{1}{5}M(\beta^2 + \gamma^2)\dot{\xi}$$

$$- \left[B(\Omega_2 + \eta) + \mu \frac{\alpha^2 - \gamma^2}{\alpha^2 + \gamma^2} \Omega_2 + \tfrac{1}{5}M(\alpha^2 + \gamma^2)\eta \right] \omega + (\Omega_2 + \eta)[C + \tfrac{1}{5}M(\alpha^2 + \beta^2)]\omega = 0$$

$$B(\dot{\Omega}_2 + \dot{\eta}) + \mu \frac{\alpha^2 - \gamma^2}{\alpha^2 + \gamma^2} \dot{\Omega}_2 + \tfrac{1}{5}M(\gamma^2 + \alpha^2)\dot{\eta}$$

$$- (\Omega_1 + \xi)[C + \tfrac{1}{5}M(\alpha^2 + \beta^2)]\omega + \left[A(\Omega_1 + \xi) + \nu \frac{\beta^2 - \gamma^2}{\beta^2 + \gamma^2} \Omega_1 + \tfrac{1}{5}M(\beta^2 + \gamma^2)\xi \right] \omega = 0$$

$$C(\dot{\Omega}_3 + \dot{\zeta}) + \tfrac{1}{5}M \frac{(\alpha^2 - \beta^2)^2}{\alpha^2 + \beta^2} \dot{\Omega}_3 + \tfrac{1}{5}M(\alpha^2 + \beta^2)\dot{\zeta} = 0.$$

HELMHOLTZ's equations of vortex motion are

$$\dot{\xi} = -\frac{2\alpha^2}{\alpha^2 + \gamma^2} \omega \Omega_2, \quad \dot{\eta} = \frac{2\beta^2}{\beta^2 + \gamma^2} \omega \Omega_1, \quad \dot{\zeta} = 0.$$

Hence, if we put ξ_1, η_1, Ω_1, Ω_2, Ω_3, $\zeta - \omega$ each proportional to $e^{i\lambda t}$, and introduce, for brevity, the notation $A' \equiv \tfrac{1}{5}M(\beta^2 + \gamma^2)$, &c.,

$$\left.\begin{aligned}
&\frac{\lambda i}{\omega}\left[\left(A + \nu \frac{\beta^2 - \gamma^2}{\beta^2 + \gamma^2}\right)\Omega_1 + (A + A')\xi\right] \\
&\qquad + \left[C + C' - B - \mu \frac{\alpha^2 - \gamma^2}{\alpha^2 + \gamma^2}\right]\Omega_2 + (C + C' - B - B')\eta = 0, \\
&\left[C + C' - A - \nu \frac{\beta^2 - \gamma^2}{\beta^2 + \gamma^2}\right]\Omega_1 \\
&\qquad + (C + C' - A - A')\xi - \frac{\lambda i}{\omega}\left[\left(B + \mu \frac{\alpha^2 - \gamma^2}{\alpha^2 + \gamma^2}\right)\Omega_2 + (B + B')\eta\right] = 0, \\
&\frac{\lambda i}{\omega}\xi + \frac{2\alpha^2}{\alpha^2 + \gamma^2}\Omega_2 = 0, \quad \frac{\lambda i}{\omega}\eta - \frac{2\beta^2}{\beta^2 + \gamma^2}\Omega_1 = 0
\end{aligned}\right\} \quad (4).$$

$$\Omega_3 = 0 \qquad \zeta = \omega.$$

Eliminating Ω_1, Ω_2, ξ, η, the period equation is

$$\begin{vmatrix}
\frac{\lambda i}{\omega}\left[A + \nu \frac{\beta^2 - \gamma^2}{\beta^2 + \gamma^2}\right], & C + C' - B - \mu \frac{\alpha^2 - \gamma^2}{\alpha^2 + \gamma^2}, & \frac{\lambda i}{\omega}(A + A'), & C + C' - B - B' \\
C + C' - A - \nu \frac{\beta^2 - \gamma^2}{\beta^2 + \gamma^2}, & \frac{-\lambda i}{\omega}\left[B + \mu \frac{\alpha^2 - \gamma^2}{\alpha^2 + \gamma^2}\right], & C + C' - A - A', & \frac{-\lambda i}{\omega}(B + B') \\
0, & \frac{2\alpha^2}{\alpha^2 + \gamma^2}, & \frac{\lambda i}{\omega}, & 0, \\
\frac{-2\beta^2}{\beta^2 + \gamma^2}, & 0, & 0, & \frac{\lambda i}{\omega}
\end{vmatrix} = 0.$$

On expanding and arranging according to powers ω, λ, this determinant reduces to

$$\begin{aligned}\lambda^4 [\mathrm{A}\,(\beta^2+\gamma^2)+\nu\,(\beta^2-\gamma^2)]\,[\mathrm{B}\,(\alpha^2+\gamma^2)+\mu\,(\alpha^2-\gamma^2)] \\ -\,\omega^2\lambda^2\,[(\alpha^2+\gamma^2)\,(\beta^2+\gamma^2)\,(\mathrm{B}-\mathrm{C}-\nu)\,(\mathrm{A}-\mathrm{C}-\mu)+4\alpha^2\beta^2\,(\mathrm{A}+\nu)\,(\mathrm{B}+\mu) \\ +\,4\mu\beta^2\gamma^2\,(\mathrm{B}-\mathrm{C}-\nu)+4\nu\alpha^2\gamma^2\,(\mathrm{A}-\mathrm{C}-\mu)] \\ +\,\omega^4\,\{4\alpha^2\beta^2\,(\mathrm{B}-\mathrm{C}-\nu)\,(\mathrm{A}-\mathrm{C}-\mu)\}=0. \quad \quad (5).\end{aligned}$$

§ 2. *Case of Shell without Inertia.*

If the shell be so thin that we may neglect its inertia compared with that of the fluid, we may put $\mathrm{A}=\mathrm{B}=\mathrm{C}=0$, and equation (5) then becomes

$$(\alpha^2-\gamma^2)\,(\beta^2-\gamma^2)\,\lambda^4-\omega^2\lambda^2\,\{\gamma^4-3\,(\alpha^2+\beta^2)\,\gamma^2+5\alpha^2\beta^2\}+4\omega^4\alpha^2\beta^2=0 \quad (6);$$

when the system is symmetrical about the axis of rotation $\alpha^2=\beta^2$, and this equation reduces to

$$\lambda^4\,(\alpha^2-\gamma^2)^2-\omega^2\lambda^2\,(\alpha^2-\gamma^2)\,(5\alpha^2-\gamma^2)+4\omega^4\alpha^2\beta^2=0,$$

the roots of which are

$$\lambda^2=\frac{\omega^2}{4}\left\{1\pm\sqrt{\left(\frac{9\alpha^2-\gamma^2}{\alpha^2-\gamma^2}\right)}\right\}^2 \quad \quad (7).$$

These are the same as the values obtained by BRYAN ('Phil. Trans.,' 1889, A, p. 208), for the case of a spheroid whose surface is free. As is there indicated, the modes of oscillation corresponding to these periods are such that the surface of the spheroid maintains its shape, but changes its position. Such oscillations will, of course, not be affected by supposing the fluid contained in a rigid shell without inertia, and we might have expected to obtain the same values for the periods, when the figure of the shell agrees with a possible figure of equilibrium of the fluid rotating freely.

From (7) we see that the roots, if real, are positive; in order that they may be real, we require that $9\alpha^2-\gamma^2$ and $\alpha^2-\gamma^2$ must have the same sign.

Hence a necessary condition for ordinary stability is

$$\gamma^2>9\alpha^2 \quad \text{or} \quad \gamma^2<\alpha^2,$$

i.e., γ must not lie between α and 3α.

Returning to the case where $\alpha^2\neq\beta^2$, in order that the roots may be real and positive, we must have

(1) $(\gamma^2-\alpha^2)\,(\gamma^2-\beta^2)>0.$
(2) $\gamma^4-3\,(\alpha^2+\beta^2)\,\gamma^2+5\alpha^2\beta^2>0.$
(3) $\{\gamma^4-3\,(\alpha^2+\beta^2)\,\gamma^2+5\alpha^2\beta^2\}^2-16\alpha^2\beta^2\,(\gamma^2-\alpha^2)\,(\gamma^2-\beta^2)>0.$

The 1st condition requires that γ^2 should not lie between α^2 and β^2.

Now

$$\gamma^4 - 3(\alpha^2 + \beta^2)\gamma^2 + 5\alpha^2\beta^2 = (\gamma^2 - 3\alpha^2)(\gamma^2 - 3\beta^2) - 4\alpha^2\beta^2$$
$$= (3\alpha^2 - \gamma^2)(3\beta^2 - \gamma^2) - 4\alpha^2\beta^2.$$

The 1st form shows that condition (2) is certainly satisfied if

$$\gamma^2 > 5\alpha^2 \quad \text{and also} \quad > 5\beta^2.$$

The 2nd shows that it is satisfied if $\gamma^2 < \alpha^2$ and $< \beta^2$.

Lastly,

$$\{\gamma^4 - 3(\alpha^2 + \beta^2)\gamma^2 + 5\alpha^2\beta^2\}^2 - 16\alpha^2\beta^2(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)$$
$$= (\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)\{\gamma^4 - 5(\alpha^2 + \beta^2)\gamma^2 + 9\alpha^2\beta^2\} + 4(\alpha^2 - \beta^2)^2\gamma^4$$
$$= \begin{cases} (\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)\{(\gamma^2 - 5\alpha^2)(\gamma^2 - 5\beta^2) - 16\alpha^2\beta^2\} + 4(\alpha^2 - \beta^2)^2\gamma^4 \\ \text{or} \\ (\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)\{(5\alpha^2 - \gamma^2)(5\beta^2 - \gamma^2) - 16\alpha^2\beta^2\} + 4(\alpha^2 - \beta^2)^2\gamma^4. \end{cases}$$

Hence condition (3) will certainly be satisfied if $\gamma^2 > 9\alpha^2$ and also $> 9\beta^2$, or if $\gamma^2 < \alpha^2$ and also $< \beta^2$.

Thus the roots of (6) will both be real and positive if

$$\gamma < \alpha \quad \text{and also} \quad < \beta,$$

or if

$$\gamma > 3\alpha \quad \text{and also} \quad > 3\beta.$$

These conditions are sufficient, but not necessary, to ensure stability; the necessary conditions are given by the inequalities (1), (2), (3).

The analytical conditions here discussed are approximately realised in the case of a liquid gyrostat (*vide* 'Nature,' vol. 15, p. 297) mounted on gimbals in such a way that the centre of gravity is held at rest. The inertia of the gimbal-rings will be unimportant when the rotation is rapid, and, if we may also neglect the inertia of the case compared with that of the fluid, the gyrostat will be stable when set rotating about its least axis; it will also be stable when set rotating about its greatest axis when this axis is, at least, three times as great as either of the others. It will, however, certainly be unstable when set rotating about its mean axis.

§ 3. *Approximate Solution of the Period Equation.*

Let us for the future suppose that the cavity which contains fluid is approximately spherical, so that $\frac{\alpha - \gamma}{\gamma}$, $\frac{\beta - \gamma}{\gamma}$ are small quantities.

Suppose

$$\frac{\alpha - \gamma}{\gamma} = \epsilon_1, \quad \frac{\beta - \gamma}{\gamma} = \epsilon_2.$$

From (3) we have

$$\nu (\alpha^2 - \gamma^2) = \mu (\beta^2 - \gamma^2);$$

therefore, if we neglect squares of ϵ_1, ϵ_2,

$$\mu/\epsilon_1 = \nu/\epsilon_2 = q\mathrm{C} \text{ say.}$$

In the case where the thickness of the shell is finite compared with its linear dimensions q will be a finite quantity; when the shell is thin q will be large, and when the fluid nucleus is small compared with the dimensions of the whole system q will be small, the densities of the fluid and of the crust being supposed comparable with one another. In all cases q will be positive.

If ϵ_1, ϵ_2 are each equal to zero, the equation (5) becomes

$$\mathrm{AB}\lambda^4 - [(\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B}) + \mathrm{AB}]\,\lambda^2\omega^2 + (\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B})\,\omega^4 = 0,$$

or

$$(\lambda^2 - \omega^2)[\mathrm{AB}\lambda^2 - (\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B})\,\omega^2] = 0.$$

Thus we obtain as a first approximation to the roots

$$\lambda^2 = \omega^2, \quad \lambda^2 = \frac{(\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B})}{\mathrm{AB}}\,\omega^2.$$

Next let us retain first powers of ϵ_1, ϵ_2 in (5); this equation then becomes

$$\begin{aligned}&\lambda^4\mathrm{AB}(1 + \epsilon_1 + \epsilon_2)\\ &\quad - \omega^2\lambda^2[(\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B})(1 + \epsilon_1 + \epsilon_2) + \mathrm{AB}(1 + 2\epsilon_1 + 2\epsilon_2) + q(\mathrm{AC}\epsilon_1 + \mathrm{BC}\epsilon_2)]\\ &\quad + \omega^4(1 + 2\epsilon_1 + 2\epsilon_2)(\mathrm{C} - \mathrm{A} + q\mathrm{C}\epsilon_1)(\mathrm{C} - \mathrm{B} + q\mathrm{C}\epsilon_2) = 0,\end{aligned}$$

or

$$\begin{aligned}&(\lambda^2 - \omega^2)(\mathrm{AB}\lambda^2 - \overline{\mathrm{C} - \mathrm{A}}\,\overline{\mathrm{C} - \mathrm{B}}\,\omega^2)\\ &\quad + \epsilon_1[\mathrm{AB}\lambda^4 - \omega^2\lambda^2\{(\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B}) + 2\mathrm{AB} + q\mathrm{AC}\}\\ &\qquad\qquad + \omega^4(\mathrm{C} - \mathrm{B})\{2(\mathrm{C} - \mathrm{A}) + q\mathrm{C}\}]\\ &\quad + \epsilon_2[\mathrm{AB}\lambda^4 - \omega^2\lambda^2\{(\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B}) + 2\mathrm{AB} + q\mathrm{BC}\}\\ &\qquad\qquad + \omega^4(\mathrm{C} - \mathrm{A})\{2(\mathrm{C} - \mathrm{B}) + q\mathrm{C}\}] = 0 \quad . \quad (8);\end{aligned}$$

dividing by $\mathrm{AB}\lambda^2 - (\mathrm{C} - \mathrm{A})(\mathrm{C} - \mathrm{B})\,\omega^2$, and putting $\lambda^2 = \omega^2$ in the terms which contain ϵ_1 or ϵ_2 as a factor, we obtain as a closer approximation to the root $\lambda^2 = \omega^2$,

$$\lambda^2 = \omega^2 . \left[1 - \epsilon_1 \left\{\frac{(C-A)(C-B) - AB + qC(C-B-A)}{AB - (C-A)(C-B)}\right\} - \epsilon_2 \left\{\frac{(C-A)(C-B) - AB + qC(C-B-A)}{AB - (C-A)(C-B)}\right\}\right]$$

$$= \omega^2 [1 + (\epsilon_1 + \epsilon_2)(1+q)] = \omega^2 . (1 + 2E) \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

therefore

$$\lambda = \pm \omega (1 + E).$$

Again, dividing by $\lambda^2 - \omega^2$ in (8), and putting $\lambda^2 = \frac{(C-A)(C-B)}{AB}\omega^2$ in the small terms, an approximate value of the second root, correct to first powers of ϵ_1, ϵ_2, is given by

$$AB\lambda^2 - (C-A)(C-B)\,\omega^2 = -\epsilon_1\omega^2 \frac{qC(C-B) - qAC\frac{(C-A).(C-B)}{AB}}{\frac{(C-A)(C-B)}{AB} - 1} - \epsilon_2\omega^2 \frac{qC(C-A) - qBC\frac{(C-A)(C-B)}{AB}}{\frac{(C-A)(C-B)}{AB} - 1},$$

or

$$\lambda^2 = \omega^2 \left[\frac{(C-A)\,(C-B)}{AB} + \epsilon_1 q \frac{C-B}{B} + \epsilon_2 q \frac{C-A}{A}\right]$$

$$= \omega^2 \left[\frac{C-A}{A} + \epsilon_1 q\right]\left[\frac{C-B}{B} + \epsilon_2 q\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

to the same order of approximation.

This approximation involves the assumption that ϵ_1, ϵ_2 are small compared with $\frac{C-A}{A}$, $\frac{C-B}{B}$; the approximate value of the root will, however, be the same if we suppose $\frac{C-A}{A}$, $\frac{C-B}{B}$ to be small quantities of the same order as ϵ_1, ϵ_2.

Let us put

$$\frac{C-A}{A} = \kappa_1, \quad \frac{C-B}{B} = \kappa_2.$$

Retaining only finite terms in (5), we obtain as a first approximation to the roots $\lambda^2 = \omega^2$ and $\lambda^2 = 0$; also, the independent term in (5) is a small quantity of the second order in κ_1, κ_2, ϵ_1, ϵ_2. Thus the root which approximates to $\lambda^2 = 0$ will be of the second order. Regarding λ^2 as of the second order, and retaining only terms of this order in (5), we get

$$\lambda^2 . [4AB\alpha^2\beta^2] - \omega^2 . 4\alpha^2\beta^2 . \{B\kappa_2 + qC\epsilon_2\}\{A\kappa_1 + qC\epsilon_1\} = 0,$$

or

$$\lambda^2 = \omega^2 . \left(\kappa_1 + q \frac{\mathrm{C}}{\mathrm{A}} \epsilon_1\right)\left(\kappa_2 + q \frac{\mathrm{C}}{\mathrm{B}} \epsilon_2\right)$$
$$= \omega^2 . (\kappa_1 + q\epsilon_1)\ (\kappa_2 + q\epsilon_2)$$

to the same order, and this is the value obtained above (10).

§ 4. *Application to the Case of the Earth.*

The nature of the two types of oscillation will be found fully discussed in the Appendix. It is there shown that the oscillation corresponding to the root $\omega\ (1 + \mathrm{E})$ is that previously examined by HOPKINS in his 'Researches in Physical Geology,' whereas the second type is analogous to the motion of a rigid body when disturbed from a motion of rotation about a principal axis.

If the Earth could be regarded as a system such as we have been considering, we see that in addition to the ordinary Solar and Lunar Nutations, which would be of the same nature as when the Earth is supposed solid throughout, there might exist certain free nutations the amplitude of which could only be determined by observation. If the amplitudes were sufficiently large, the oscillations corresponding to the root $\lambda = \omega\ (1 + \mathrm{E})$ would render themselves visible in the same way as the Solar and Lunar Nutations, namely, by small periodic displacements common to all stars. The period of these displacements would be $1/\mathrm{E}$ sidereal days, and a knowledge of it would enable us to determine E, a quantity which depends on the form of the internal surface and the thickness of the crust.

The oscillations which correspond to the root $\lambda = \omega\sqrt{\{(\kappa_1 + q\epsilon_1)\ (\kappa_2 + q\epsilon_2)\}}$ would manifest themselves in a different manner. They are, in fact, similar to the "Eulerian" nutation (*vide* TISSERAND, 'Mécanique Céleste,' vol. 2, p. 494), and will involve a small periodic change in the latitude of places on the Earth's surface, as found by meridian observations of a circumpolar star, this change taking place in a period of $\{(\kappa_1 + q\epsilon_1)\ (\kappa_2 + q\epsilon_2)\}^{-\frac{1}{2}}$ sidereal days.

Now it appears probable that in oscillations of long period, such as Precession, the effects of fluid friction would be to make the internal fluid move with the crust as if rigidly connected to it (TISSERAND, 'Mécanique Céleste,' vol. 2, p. 480, or Lord KELVIN, 'Popular Lectures and Addresses,' vol. 2, p. 244). Hence, if $\mathfrak{A}$, $\mathfrak{A}$, $\mathfrak{C}$ be the principal moments of inertia for the Earth as a whole, supposed symmetrical about its axis of rotation, the Theory of Precession will still enable us to determine the value of $\dfrac{\mathfrak{C} - \mathfrak{A}}{\mathfrak{C}}$, as $\frac{1}{305}$.

But, if we put $\kappa_2 = \kappa_1$, $\epsilon_2 = \epsilon_1$, and denote by M the mass of the fluid,

$$\mathfrak{C} = \mathrm{C} + \tfrac{2}{5}\mathrm{M}\alpha^2 = \mathrm{C}\ \{1 + q\ (1 + 2\epsilon_1)\},$$
$$\mathfrak{A} = \mathrm{A} + \tfrac{1}{5}\mathrm{M}\ (\alpha^2 + \gamma^2) = \mathrm{A} + q\mathrm{C}\ (1 + \epsilon_1).$$

Therefore,

$$\frac{\mathfrak{C} - \mathfrak{A}}{\mathfrak{C}} = \frac{C - A + qC\epsilon_1}{C(1 + q + 2q\epsilon_1)} = \frac{\kappa_1 + q\epsilon_1}{1 + q}$$

very nearly.

Therefore the period in which the latitude variations will take place is

$$\frac{1}{\kappa_1 + q\epsilon_1} \text{ sidereal days} = \frac{\mathfrak{C}}{\mathfrak{C} - \mathfrak{A}} \cdot \frac{1}{1 + q} \text{ sidereal days};$$

when the Earth is supposed solid throughout, this period is

$$\frac{\mathfrak{C}}{\mathfrak{C} - \mathfrak{A}} \text{ sidereal days.}$$

We thus see that if the Earth consisted of a rigid shell containing a homogeneous fluid nucleus, the theoretical period of 305 days, calculated on the assumption of the Earth's rigidity throughout, would be diminished in the ratio $1 : 1 + q$, where q is an essentially positive quantity, whose magnitude increases with the size of the nucleus.

In order to form some idea of the magnitude of this effect, let us suppose that the fluid and the crust have the same density ρ, and that r, r_1 are the mean radii of the fluid nucleus, and of the Earth as a whole.

We then have approximately

$$\mu = \nu = \tfrac{8}{15}\pi\rho r^5\epsilon_1 \text{ and } C = \tfrac{8}{15}\pi\rho(r_1^5 - r^5).$$

Therefore,

$$q = \frac{\mu}{\epsilon_1 C} = \frac{r^5}{r_1^5 - r^5},$$

$$1 + q = \frac{r_1^5}{r_1^5 - r^5}, \qquad \frac{1}{1+q} = 1 - \left(\frac{r}{r_1}\right)^5,$$

and the period will be diminished by $\left(\frac{r}{r_1}\right)^5 \times 305$ days.

Taking the mean radius of the Earth as 4000 miles, we obtain the following table, where the first line gives the thickness of the crust in miles, and the second the diminution of the period in sidereal days :—

Thickness of crust in miles . . .	2000	1000	500	250	100
Diminution of period in days . .	5	21	156	221	269

Now, although in the light of Professor NEWCOMB'S work ('Astron. Soc. Monthly Notices,' March, 1892), it appears probable that this effect would be modified by the elasticity of the Crust, it could scarcely be reversed if the fluid nucleus were of any considerable extent. We must, therefore, conclude that the observations on latitude-variation, so far from establishing the existence of a fluid interior, as supposed by M. FOLIE, rather tend to confirm the views hitherto maintained by physicists on other grounds, that there can be no internal fluid mass of any considerable extent.

APPENDIX.

TREATMENT OF THE PROBLEM BY LAMÉ ANALYSIS.

§ 1. *Equations of Motion of Fluid.*

Let us refer to rectangular axes rotating with angular velocity ω about the axis of z. The fluid is supposed to have no motion relatively to these axes other than that due to the small oscillations with which we are dealing.

Let u, v, w, be the velocity-components at any point x, y, z relatively to these axes: we shall, as is usual in small-oscillation problems, neglect squares and products of the small quantities u, v, w.

The actual velocity-components parallel to the instantaneous positions of the moving axes will be

$$u - \omega y, \quad v + \omega x, \quad w,$$

and the differential equations of motion of the fluid are therefore (BASSET, 'Hydrodynamics,' p. 22)

$$\frac{\partial u}{\partial t} - \omega(v + \omega x) - \omega v = \frac{\partial}{\partial x}\left(\mathrm{V}_1 - \frac{p}{\rho_1}\right),$$
$$\frac{\partial v}{\partial t} + \omega(u - \omega y) + \omega u = \frac{\partial}{\partial y}\left(\mathrm{V}_1 - \frac{p}{\rho_1}\right),$$
$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial z}\left(\mathrm{V}_1 - \frac{p}{\rho_1}\right),$$

where V_1 is the gravitation-potential of the forces to which the fluid is subject, p the fluid-pressure, and ρ_1 the density.

Putting

$$\psi = \mathrm{V}_1 - p/\rho_1 + \tfrac{1}{2}\omega^2(x^2 + y^2) \quad \ldots \ldots \quad (1),$$

the above equations reduce to

$$\left.\begin{aligned} \partial u/\partial t - 2\omega v &= \partial\psi/\partial x \\ \partial v/\partial t + 2\omega u &= \partial\psi/\partial y \\ \partial w/\partial t &= \partial\psi/\partial z \end{aligned}\right\} \quad \ldots \ldots \quad (2).$$

We have, in addition, the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Equations (2), (3) are sufficient to determine u, v, w, ψ, subject to certain boundary conditions.

From (2) we obtain

$$\left.\begin{aligned} \left[\frac{\partial^2}{\partial t^2} + 4\omega^2\right] u &= \frac{\partial^2 \psi}{\partial x \partial t} + 2\omega \frac{\partial \psi}{\partial y} \\ \left[\frac{\partial^2}{\partial t^2} + 4\omega^2\right] v &= \frac{\partial^2 \psi}{\partial y \partial t} - 2\omega \frac{\partial \psi}{\partial x} \\ \frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2 \psi}{\partial z \partial t} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Applying the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, and adding, we obtain by means of (3)

$$-4\omega^2 \frac{\partial w}{\partial z} = \frac{\partial}{\partial t}(\nabla^2 \psi), \text{ where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

or, by the third of equations (2),

$$4\omega^2 \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2}{\partial t^2}(\nabla^2 \psi) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

This is POINCARÉ's differential equation for the oscillations of a mass of fluid about a steady motion of pure rotation.

Let us now suppose that the system is executing one of its component harmonic vibrations.

Assume that

$$u = u_1 \epsilon^{i\lambda t}, \qquad v = v_1 \epsilon^{i\lambda t}, \qquad w = w_1 \epsilon^{i\lambda t},$$

and

$$\psi = \psi_1 \epsilon^{i\lambda t}.$$

Putting these values in (4), (5), and dividing out by the time factor, we get

$$\left.\begin{aligned} u_1 &= \frac{1}{4\omega^2 - \lambda^2}\left\{ i\lambda \frac{\partial \psi_1}{\partial x} + 2\omega \frac{\partial \psi_1}{\partial y}\right\} \\ v_1 &= \frac{1}{4\omega^2 - \lambda^2}\left\{ i\lambda \frac{\partial \psi_1}{\partial y} - 2\omega \frac{\partial \psi_1}{\partial x}\right\} \\ w_1 &= \frac{1}{i\lambda} \frac{\partial \psi_1}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

while ψ_1 satisfies the equation

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} + \left(1 - \frac{4\omega^2}{\lambda^2}\right) \frac{\partial^2 \psi_1}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

§ 2. *The Boundary Conditions.*

The position of the shell at any instant may be defined by means of three coordinates, θ_1, θ_2, θ_3, which denote the small angular displacements, about the axes of reference, of the shell from the position it would occupy in the steady motion.

The displacements parallel to the coordinate axes of the point of the shell, whose coordinates are (x, y, z), are

$$-y\theta_3 + z\theta_2, \qquad -z\theta_1 + x\theta_3, \qquad -x\theta_2 + y\theta_1.$$

If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines of the normal to the undisturbed surface, the normal distance between this surface and the displaced surface will be

$$\begin{aligned}(-y\theta_3 + z\theta_2)\cos\alpha + (-z\theta_1 + x\theta_3)\cos\beta + (-x\theta_2 + y\theta_1)\cos\gamma \\ = \theta_1(y\cos\gamma - z\cos\beta) + \theta_2(z\cos\alpha - x\cos\gamma) + \theta_3(x\cos\beta - y\cos\alpha).\end{aligned}$$

The condition to be satisfied at the boundary is that the rate of increase of this length must be equal to the component velocity of the fluid, relative to the moving axes, in the direction of the normal to the undisturbed surface. Now as these relative velocities are all small quantities whose squares we are neglecting, it is unnecessary to distinguish between the velocities at the disturbed and undisturbed surfaces; thus, at the latter surface we require

$$\begin{aligned}u\cos\alpha + v\cos\beta + w\cos\gamma = \dot\theta_1(y\cos\gamma - z\cos\beta) \\ + \dot\theta_2(z\cos\alpha - x\cos\gamma) + \dot\theta_3(x\cos\beta - y\cos\alpha).\end{aligned}$$

Putting $\theta_1 = \theta'_1 e^{i\lambda t}$, &c., and omitting the exponential factor, we obtain

$$\begin{aligned}u_1\cos\alpha + v_1\cos\beta + w_1\cos\gamma = i\lambda\,[\theta'_1(y\cos\gamma - z\cos\beta) \\ + \theta'_2(z\cos\alpha - x\cos\gamma) + \theta'_3(x\cos\beta - y\cos\alpha)];\end{aligned}$$

or, putting in the values of u_1, v_1, w_1, ψ_1 from (6),

$$\begin{aligned}\frac{\lambda}{4\omega^2 - \lambda^2}\left\{\frac{\partial\psi_1}{\partial x}\cos\alpha + \frac{\partial\psi_1}{\partial y}\cos\beta + \frac{\partial\psi_1}{\partial z}\cos\gamma\left(1 - \frac{4\omega^2}{\lambda^2}\right)\right\} \\ - \frac{2\omega i}{4\omega^2 - \lambda^2}\left\{\frac{\partial\psi_1}{\partial y}\cos\alpha - \frac{\partial\psi_1}{\partial x}\cos\beta\right\} \\ = \lambda[\theta'_1(y\cos\gamma - z\cos\beta) + \theta'_2(z\cos\alpha - x\cos\gamma) \\ + \theta'_3(x\cos\beta - y\cos\alpha)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).\end{aligned}$$

Let us now put

$$1 - \frac{4\omega^2}{\lambda^2} = \tau^2, \quad z = \tau z'.$$

Corresponding to any series of points whose coordinates are denoted by (x, y, z), we shall obtain a new series whose coordinates are (x, y, z'), which will be real or imaginary according as λ^2 is greater or less than $4\omega^2$. We will take as the standard case that in which $\lambda^2 > 4\omega^2$.

If the point (x, y, z) trace out any surface whose equation is $f(x, y, z) = 0$, the corresponding point (x, y, z') will trace out a new surface whose equation is $f(x, y, \tau z') = 0$. The part of this latter surface which corresponds to the real part of the surface $f(x, y, z) = 0$ will, however, be purely imaginary if $\lambda^2 < 4\omega^2$.

If $(\cos\alpha, \cos\beta, \cos\gamma)$, $(\cos\alpha', \cos\beta', \cos\gamma')$ be the direction-cosines of the normals to the two surfaces, we have

$$\begin{aligned}\cos\alpha' : \cos\beta' : \cos\gamma' &= \partial f/\partial x : \partial f/\partial y : \partial f/\partial z' \\ &= \partial f/\partial x : \partial f/\partial y : \tau\,(\partial f/\partial z) \\ &= \cos\alpha : \cos\beta : \tau\cos\gamma.\end{aligned}$$

Substituting in equations (7), (8), the differential equation for ψ_1 takes the form

$$\partial^2\psi_1/\partial x^2 + \partial^2\psi_1/\partial y^2 + \partial^2\psi_1/\partial z'^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

while if $f(x, y, z) = 0$ be the equation to the undisturbed boundary of the fluid, ψ_1 must satisfy the equation

$$\lambda . \left\{\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z_1}\cos\gamma'\right\} - 2\omega i\left\{\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta'\right\}$$

$$= (4\omega^2 - \lambda^2)\,\lambda . \left[\begin{array}{l} \theta_1'\left(\dfrac{y}{\tau}\cos\gamma' - \tau z'\cos\beta'\right) \\ + \theta_2'\left(\tau z'\cos\alpha' - \dfrac{x}{\tau}\cos\gamma'\right) \\ + \theta_3'\,(x\cos\beta' - y\cos\alpha') \end{array}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

at the surface $f(x, y, z') = 0$.

The problem of finding the motion of the fluid is thus reduced to that of obtaining solutions of equation (9) consistent with the boundary condition (10) at the surface $f(x, y, \tau z') = 0$.

§ 3. *Case of Ellipsoidal Surface.*

Hitherto, no assumption has been made as to the form of the surface of the fluid. Let us now suppose that it is given by the equation

$$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$

that is to say, that it is an ellipsoid whose principal axes coincide with the axes of reference.

We will take as the standard case, that in which

$$\rho^2 > c^2 > b^2 > 0.$$

Put

$$\rho^2 - c^2 = (\rho^2 - c'^2)\,\tau^2.$$

The equation to the auxiliary surface $f(x, y, \tau z') = 0$ becomes

$$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z'^2}{\rho^2 - c'^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}').$$

Let us now introduce two sets of elliptic coordinates (ρ, μ, ν), (ρ', μ', ν'), connected with (x, y, z) (x, y, z') respectively by the equations

$$\frac{x}{\rho} = \frac{\mu\nu}{bc}, \quad \frac{y}{\sqrt{(\rho^2 - b^2)}} = \frac{\sqrt{(\mu^2 - b^2)(b^2 - \nu^2)}}{b\sqrt{(c^2 - b^2)}}, \quad \frac{z}{\sqrt{(\rho^2 - c^2)}} = \frac{\sqrt{(c^2 - \mu^2)(c^2 - \nu^2)}}{c\sqrt{(c^2 - b^2)}} \quad . \quad (11),$$

$$\frac{x}{\rho'} = \frac{\mu'\nu'}{bc'}, \quad \frac{y}{\sqrt{(\rho'^2 - b^2)}} = \frac{\sqrt{(\mu'^2 - b^2)(b^2 - \nu'^2)}}{b\sqrt{(c'^2 - b^2)}}, \quad \frac{z'}{\sqrt{(\rho'^2 - c'^2)}} = \frac{\sqrt{(c'^2 - \mu'^2)(c'^2 - \nu'^2)}}{c'\sqrt{(c'^2 - b^2)}} \quad (12),$$

ρ' will be equal to ρ for points which lie on the surfaces (A) (A′), but not otherwise.

Let us also put

$$\mathrm{X} = x/\rho, \quad \mathrm{Y} = y/\sqrt{(\rho^2 - b^2)}, \quad \mathrm{Z} = z/\sqrt{(\rho^2 - c^2)} = z'/\sqrt{(\rho'^2 - c'^2)}, \quad . \quad (13)$$

for points on these surfaces; so that X, Y, Z are subject to the relation

$$\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B}).$$

X, Y, Z may therefore be regarded as the coordinates of a point lying on a sphere of unit radius.

Denote by R, M, N three conjugate Lamé functions of the elliptic coordinates ρ, μ, ν, and by R′, M′, N′ three similar functions of the coordinates ρ', μ', ν'.

A form of solution of equation (9) convenient for satisfying boundary conditions at the surface (A′) is

$$\psi_1 = \Sigma \mathrm{A'R'M'N'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

The effect of the fluid on the motion of the shell will depend only on the fluid pressure over the surface, and this by (1) will involve the value of ψ_1 at the surface.

To find the value of ψ_1 at the surface (A), we may transform the expression (14) for ψ_1 first to the surface (B) and then from (B) to (A).

Now, by a known property of Lamé products (*vide* HEINE, 'Kugelfunctionen,' vol. 1, § 89), if M, N be two conjugate functions of order n, the product MN at the surface (A) will transform into a surface harmonic of order n at the surface (B); and, conversely, any surface harmonic of order n at the surface (B), when transformed to the surface (A), can be expanded in a series consisting of Lamé products with constant coefficients, each of which products will be of the n^{th} order.

The same conclusions will hold for the surface (A′) and the sphere (B).

We can thus express the value of ψ_1 at the surface (A) in terms of a series of Lamé products, in which each term will be of the same order as that from which it arises in (14).

The couples on the shell due to fluid pressure are

$$\iint p\,d\sigma\,(y\cos\gamma - z\cos\beta), \quad \iint p\,d\sigma\,(z\cos\gamma - x\cos\beta), \quad \iint p\,d\sigma\,(x\cos\beta - y\cos\gamma),$$

where $d\sigma$ is an element of the surface and the integrals are taken over the whole surface.

If P denote the perpendicular from the centre on the tangent plane to the ellipsoid (A) and $l = \dfrac{\mathrm{P}}{\rho\sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}}$

$$y\cos\gamma - z\cos\beta = \mathrm{P}yz\left\{\frac{1}{\rho^2 - c^2} - \frac{1}{\rho^2 - b^2}\right\} = \frac{\mathrm{P}yz\,(c^2 - b^2)}{(\rho^2 - c^2)(\rho^2 - b^2)},$$

and $\mathrm{P}yz$ is proportional to

$$l\sqrt{(\mu^2 - b^2)(c^2 - \mu^2)}\,\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)} = l\mathrm{M}_1\mathrm{N}_1,$$

where M_1, N_1 are two conjugate Lamé functions of the second order. But, if MN be any two conjugate Lamé functions different from $\mathrm{M}_1\mathrm{N}_1$, $\iint l\mathrm{MNM}_1\mathrm{N}_1\,d\sigma = 0$. For, if we transform to the surface of the sphere (B), $l\,d\sigma$ is equal to the corresponding element of the spherical surface, and MN, $\mathrm{M}_1\mathrm{N}_1$ transform into two different surface harmonics.

Thus the only term in ψ_1 which can give rise to any couple about the axis of x will be the term involving the Lamé product $\mathrm{M}_1\mathrm{N}_1$.

Similarly the terms which can give rise to couples about the other axes will be of the second order. These, as we have seen above, all arise from terms of the second order in (14), and, in order to evaluate these couples, it will be unnecessary for us to calculate any coefficients in ψ_1 other than those of terms of the second order.

§ 4. *Transformation of Boundary Equations.*

Let us now transform our boundary conditions to the surface of the sphere (B).

We can at once express the right-hand member of equation (10) in terms of X, Y, Z; for

$$\cos\alpha' = \frac{\mathrm{P}'x}{\rho^2} = \frac{\mathrm{P}'\mathrm{X}}{\rho},$$

$$\cos\beta' = \frac{\mathrm{P}'y}{\rho^2 - b^2} = \frac{\mathrm{P}'\mathrm{Y}}{\sqrt{(\rho^2 - b^2)}},$$

$$\cos\gamma' = \frac{\mathrm{P}'z'}{\rho^2 - c'^2} = \frac{\mathrm{P}'\mathrm{Z}}{\sqrt{(\rho^2 - c'^2)}},$$

where P′ has the same signification with reference to the ellipsoid (A′) as P has with reference to the ellipsoid (A).

The right-hand member therefore becomes

$$-\lambda^3\tau^2 . \mathrm{P}' . \left[\theta'_1 \frac{c^2 - b^2}{\sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}} \mathrm{YZ} + \theta'_2 \frac{(-c^2)}{\rho\sqrt{(\rho^2 - c^2)}} \mathrm{ZX} + \theta'_3 \frac{b^2}{\rho\sqrt{(\rho^2 - b^2)}} \mathrm{XY}\right] \quad (15).$$

Consider next a single term of order n in ψ_1, say $\psi_1 = \mathrm{R'M'N'}$. We have seen that M′N′ is expressible in the form $\epsilon\mathrm{S}_n$, where ϵ is some quantity which does not vary over the surface of the sphere (B) and S_n is a spherical harmonic function of degree n in X, Y, Z.

If dn' denote an element of the normal to the surface (A′), we have $\mathrm{P}'\,dn' = \rho'\,d\rho'$, and therefore,

$$\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z'}\cos\gamma' = \frac{\partial\psi_1}{\partial n'} = \frac{\mathrm{P}'}{\rho'}\frac{\partial\psi_1}{\partial\rho'}.$$

Now M′, N′ are independent of ρ'. Therefore, when $\psi_1 = \mathrm{R'M'N'}$,

$$\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z'}\cos\gamma' = \frac{\mathrm{P}'}{\rho'}\frac{\partial\mathrm{R}'}{\partial\rho'}\mathrm{M'N'} = \frac{\mathrm{P}'}{\rho'}\frac{\partial\mathrm{R}'}{\partial\rho'} . \epsilon\mathrm{S}_n \quad . \quad (16).$$

Next let ds be an element of a line through (x, y, z') whose direction-cosines are $\left(-\frac{\cos\beta'}{\sin\gamma'}, \frac{\cos\alpha'}{\sin\gamma'}, 0\right)$, and which, therefore, lies in the surface (A′).

Then

$$\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta' = \sin\gamma'\frac{\partial\psi_1}{\partial s},$$

and when $\psi_1 = \mathrm{R'M'N'} = \mathrm{R}'\epsilon\mathrm{S}_n$,

$$\frac{\partial\psi_1}{\partial s} = \frac{\partial}{\partial s}(\mathrm{R}'\epsilon) . \mathrm{S}_n + \mathrm{R}'\epsilon . \frac{\partial\mathrm{S}_n}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Now, the element ds lies entirely in the surface (A′), and corresponding to every point of it there will be a point X, Y, Z which lies on the sphere (B). Hence, as R′, ϵ do not vary over the surface of the sphere, they will remain constant as we pass along the element ds. Thus we have

$$\frac{\partial}{\partial s}(\mathrm{R}'\epsilon) = 0;$$

and, therefore, from (17)

$$\frac{\partial \psi_1}{\partial s} = \mathrm{R}'\epsilon . \frac{\partial \mathrm{S}_n}{\partial s} = \mathrm{R}'\epsilon\left\{\frac{\partial \mathrm{S}_n}{\partial \mathrm{X}} \cdot \frac{\partial \mathrm{X}}{\partial s} + \frac{\partial \mathrm{S}_n}{\partial \mathrm{Y}} \cdot \frac{\partial \mathrm{Y}}{\partial s} + \frac{\partial \mathrm{S}_n}{\partial \mathrm{Z}} \cdot \frac{\partial \mathrm{Z}}{\partial s}\right\}.$$

But from (13) we see that

$$\frac{\partial \mathrm{X}}{\partial s} = \frac{1}{\rho}\frac{\partial x}{\partial s} = \frac{1}{\rho}\left(-\frac{\cos\beta'}{\sin\gamma'}\right) = -\frac{\mathrm{P}'y}{\rho(\rho^2 - b^2)}\operatorname{cosec}\gamma' = -\frac{\mathrm{P}'\mathrm{Y}}{\rho\sqrt{(\rho^2-b^2)}}\operatorname{cosec}\gamma'$$

$$\frac{\partial \mathrm{Y}}{\partial s} = \frac{1}{\sqrt{(\rho^2 - b^2)}}\frac{\partial y}{\partial s} = \frac{1}{\sqrt{(\rho^2 - b^2)}}\left(\frac{\cos\alpha'}{\sin\gamma'}\right) = \frac{\mathrm{P}'x}{\rho^2\sqrt{(\rho^2 - b^2)}}\operatorname{cosec}\gamma' = \frac{\mathrm{P}'\mathrm{X}}{\rho\sqrt{(\rho^2 - b^2)}}\operatorname{cosec}\gamma'$$

$$\frac{\partial \mathrm{Z}}{\partial s} = \frac{1}{\sqrt{(\rho^2 - c'^2)}}\frac{\partial z'}{\partial s} = 0.$$

Therefore,

$$\frac{\partial \psi_1}{\partial s} = \frac{\mathrm{P}'.\mathrm{R}'\epsilon}{\rho\sqrt{(\rho^2 - b^2)}}\left\{\mathrm{X}\frac{\partial \mathrm{S}_n}{\partial \mathrm{Y}} - \mathrm{Y}\frac{\partial \mathrm{S}_n}{\partial \mathrm{X}}\right\}\operatorname{cosec}\gamma',$$

and

$$\frac{\partial \psi_1}{\partial y}\cos\alpha' - \frac{\partial \psi_1}{\partial x}\cos\beta' = \sin\gamma'\frac{\partial \psi_1}{\partial s} = \frac{\mathrm{P}'.\mathrm{R}'\epsilon}{\rho\sqrt{(\rho^2 - b^2)}}\left\{\mathrm{X}\frac{\partial \mathrm{S}_n}{\partial \mathrm{Y}} - \mathrm{Y}\frac{\partial \mathrm{S}_n}{\partial \mathrm{X}}\right\} \quad . \quad (18).$$

From (15), (16), (18) we see that the boundary equation, on the assumption of the form (14) for ψ_1, takes the form

$$\lambda . \left\{\Sigma \mathrm{A}'\frac{\partial \mathrm{R}'}{\partial \rho'}\epsilon \mathrm{S}_n\right\} - 2\omega i\left\{\Sigma \mathrm{A}'\frac{\mathrm{R}'\epsilon}{\rho^2 - b^2}\left(\mathrm{X}\frac{\partial \mathrm{S}_n}{\partial \mathrm{Y}} - \mathrm{Y}\frac{\partial \mathrm{S}_n}{\partial \mathrm{X}}\right)\right\}$$

$$= -\lambda^3\tau^2\left\{\theta'_1\frac{\rho(c^2 - b^2)}{\sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}}\mathrm{YZ} + \theta'_2\frac{(-c^2)}{\sqrt{(\rho^2 - c^2)}}\mathrm{ZX} + \theta'_3\frac{b^2}{\sqrt{(\rho^2 - b^2)}}\mathrm{XY}\right\} \quad (19).$$

Now the function $\mathrm{X}\frac{\partial \mathrm{S}_n}{\partial \mathrm{Y}} - \mathrm{Y}\frac{\partial \mathrm{S}_n}{\partial \mathrm{X}}$ is itself a spherical harmonic function of order n, and both S_n and $\mathrm{X}\frac{\partial \mathrm{S}_n}{\partial \mathrm{Y}} - \mathrm{Y}\frac{\partial \mathrm{S}_n}{\partial \mathrm{X}}$ may be expressed as linear functions of the $2n + 1$ independent harmonics of the nth order.

Omitting for the present the terms of the second order, if we equate to zero the coefficients of the $2n + 1$ independent harmonics, we shall obtain a series of $2n + 1$ linear equations connecting the $2n + 1$ quantities A' which occur with Lamé products of order n. These equations show that all the quantities A' vanish except for certain values of λ which satisfy the determinantal equation obtained by eliminating them. The roots of this equation will determine the possible periods of free oscillation, and when the system is oscillating in the mode corresponding to any one of these roots, Lamé products of one order (n) only will appear in ψ_1.

These modes of oscillation do not involve any motion of the shell, and it is evident that they could not be generated or destroyed by any disturbance communicated to the shell, if the fluid be free from viscosity.

We proceed now to examine more closely the modes which depend on terms of the second order. As we have seen above, terms of different orders correspond to different fundamental modes; and therefore we may for the future suppose that the second order terms alone exist in ψ_1.

§ 5. *Lamé Functions of Second Order.*

A Lamé function of order n is a function R of one of the four forms

$$R = P_n,\quad R = \sqrt{\rho^2 - b^2}\,.\,P_{n-1},\quad R = \sqrt{\rho^2 - c^2}\,.\,P_{n-1},\quad R = \sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}\,.\,P_{n-2},$$

where P_n denotes a rational, integral, algebraic function of ρ of degree n, and R satisfies the differential equation

$$(\rho^2 - b^2)(\rho^2 - c^2)\frac{d^2R}{d\rho^2} + \rho(2\rho^2 - b^2 - c^2)\frac{dR}{d\rho} = [n(n+1)\rho^2 - B]\,R \quad . \quad (20),$$

where B is a suitably chosen constant.

The Lamé functions of the second order are, therefore, the three functions

$$\rho\sqrt{\rho^2 - b^2},\qquad \rho\sqrt{\rho^2 - c^2},\qquad \sqrt{(\rho^2 - b^2)(\rho^2 - c^2)} \quad . \quad . \quad . \quad (21),$$

together with two functions of the form $\rho^2 + \beta$ . . . . (22).

To find these latter functions, substitute in equation (20) with $n = 2$; we obtain

$$2(\rho^2 - b^2)(\rho^2 - c^2) + 2\rho^2\,.\,(2\rho^2 - b^2 - c^2) \equiv (6\rho^2 - B)(\rho^2 + \beta).$$

Equating coefficients of ρ^2, and the terms independent of ρ, we get

$$-4(b^2 + c^2) = 6\beta - B,\qquad 2b^2c^2 = -\beta B.$$

Eliminating B, the values of β are given by

$$3\beta^2 + 2(b^2 + c^2)\beta + b^2c^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

Let us now apply the formulæ (16), (18), to the different forms of R′; take first $R' = \rho'\sqrt{(\rho'^2 - b^2)}$.

At the surface

$$M'N' = \mu'\sqrt{(\mu'^2 - b^2)} \,.\, \nu'\sqrt{(b^2 - \nu'^2)} = \frac{b^2c'\sqrt{(c'^2 - b^2)}}{\rho\sqrt{(\rho^2 - b^2)}}\, xy = b^2c'\sqrt{(c'^2 - b^2)} \,.\, XY.$$

Therefore, when $\psi_1 = R'M'N'$,

from (16)

$$\left.\begin{aligned} &\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z'}\cos\gamma' = P' \,.\, \frac{2\rho^2 - b^2}{\rho\sqrt{(\rho^2 - b^2)}}\, b^2c'\sqrt{(c'^2 - b^2)} \,.\, XY \\ &\text{from (18)} \\ &\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta' = P'\, b^2c'\sqrt{(c'^2 - b^2)} \,.\, (X^2 - Y^2) \end{aligned}\right\} \quad (24).$$

Similarly, when $R' = \rho'\sqrt{(\rho'^2 - c'^2)}$,

$$M'N' = \quad bc'^2\sqrt{(c'^2 - b^2)}\, XZ \quad \text{at the surface,}$$

and

$$\left.\begin{aligned} &\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z'}\cos\gamma' = P'\, \frac{2\rho^2 - c'^2}{\rho\sqrt{(\rho^2 - c'^2)}}\, bc'^2\sqrt{(c'^2 - b^2)}\, XZ, \\ &\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta' = P'\sqrt{\frac{\rho^2 - c'^2}{\rho^2 - b^2}}\,(-\, YZ) \end{aligned}\right\} (25);$$

and when $R' = \sqrt{(\rho'^2 - b^2)(\rho'^2 - c'^2)}$,

$$M'N' = bc'(c'^2 - b^2)\, YZ,$$

$$\left.\begin{aligned} &\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z'}\cos\gamma' = P'\, \frac{2\rho^2 - b^2 - c'^2}{\sqrt{(\rho^2 - b^2)(\rho^2 - c'^2)}}\, bc'(c'^2 - b^2)\, YZ, \\ &\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta' = P'\, \frac{\sqrt{(\rho^2 - c'^2)}}{\rho}\, bc'(c'^2 - b^2)\, XZ \end{aligned}\right\} (26).$$

Take, now, the form (22).

When

$$R = (\rho^2 + \beta), \quad RMN = (\rho^2 + \beta)(\mu^2 + \beta)(\nu^2 + \beta);$$

but ρ^2, μ^2, ν^2 are the three roots of the equation in α

$$\frac{x^2}{\alpha} + \frac{y^2}{\alpha - b^2} + \frac{z^2}{\alpha - c^2} = 1,$$

therefore,

$$(\alpha - \rho^2)(\alpha - \mu^2)(\alpha - \nu^2) \equiv \alpha(\alpha - b^2)(\alpha - c^2) - x^2(\alpha - b^2)(\alpha - c^2) \\ - y^2\alpha(\alpha - c^2) - z^2\alpha(\alpha - b^2);$$

putting $\alpha = -\beta$ in this identity, we obtain

$$\text{RMN} = (\beta + \rho^2)(\beta + \mu^2)(\beta + \nu^2) = \beta(\beta + b^2)(\beta + c^2) \\ + x^2(\beta + b^2)(\beta + c^2) + y^2\beta(\beta + c^2) + z^2\beta(\beta + b^2),$$

and, therefore, at the surface

$$\begin{aligned}\text{RMN} &= (\text{X}^2 + \text{Y}^2 + \text{Z}^2)\beta(\beta + b^2)(\beta + c^2) + \rho^2\text{X}^2.(\beta + b^2)(\beta + c^2) \\ &\quad + (\rho^2 - b^2)\text{Y}^2.\beta(\beta + c^2) + (\rho^2 - c^2)\text{Z}^2.\beta(\beta + b^2) \\ &= (\rho^2 + \beta)[(\beta + b^2)(\beta + c^2)\text{X}^2 + \beta(\beta + c^2)\text{Y}^2 + \beta(\beta + b^2)\text{Z}^2],\end{aligned}$$

or, since by (23) $(\beta + b^2)(\beta + c^2) = -\beta(\beta + c^2) - \beta(\beta + b^2)$,

$$\text{RMN} = (\rho^2 + \beta)[\beta(\beta + c^2)(\text{Y}^2 - \text{X}^2) + \beta(\beta + b^2)(\text{Z}^2 - \text{X}^2)],$$

therefore, when $\text{R}' = \rho'^2 + \beta'$,

$$\text{M}'\text{N}' = \beta'(\beta' + c'^2)(\text{Y}^2 - \text{X}^2) + \beta'(\beta' + b^2)(\text{Z}^2 - \text{X}^2),$$

and if $\psi_1 = \text{R}'\text{M}'\text{N}'$

$$\frac{\partial\psi_1}{\partial x}\cos\alpha' + \frac{\partial\psi_1}{\partial y}\cos\beta' + \frac{\partial\psi_1}{\partial z}\cos\gamma' \\ = 2\text{P}'[\beta'(\beta' + c'^2)(\text{Y}^2 - \text{X}^2) + \beta'(\beta' + b^2)(\text{Z}^2 - \text{X}^2)] \quad . \quad (27),$$

and

$$\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta' = \text{P}'\frac{\rho^2 + \beta'}{\rho\sqrt{(\rho^2 - b^2)}}2\text{XY}\{2\beta'(\beta' + c'^2) + \beta'(\beta' + b^2)\},$$

or since β' satisfies an equation similar to (23),

$$\frac{\partial\psi_1}{\partial y}\cos\alpha' - \frac{\partial\psi_1}{\partial x}\cos\beta' = -\text{P}'\frac{\rho^2 + \beta'}{\rho\sqrt{(\rho^2 - b^2)}}2\text{XY}\{(c'^2 + \beta')b^2\} \quad . \quad . \quad (28).$$

Let us denote by β'_1, β'_2, the two values of β', and assume that

$$
\begin{aligned}
\psi_1 = {} & A_1 (\rho'^2 + \beta'_1)(\mu'^2 + \beta'_1)(\nu'^2 + \beta'_1) + A_2 (\rho'^2 + \beta'_2)(\mu'^2 + \beta'_2)(\nu'^2 + \beta'_2) \\
& + \frac{B_1}{b^2 c' \sqrt{(c'^2 - b^2)}} \rho' \sqrt{(\rho'^2 - b^2)} . \mu' \sqrt{(\mu'^2 - b^2)} . \nu' \sqrt{(b^2 - \nu'^2)} \\
& + \frac{B_2}{bc'^2 \sqrt{(c'^2 - b^2)}} \rho' \sqrt{(\rho'^2 - c'^2)} . \mu' \sqrt{(c'^2 - \mu'^2)} . \nu' \sqrt{(c'^2 - \nu'^2)} \\
& + \frac{B_3}{bc' (c'^2 - b^2)} \sqrt{(\rho'^2 - b^2)(\rho'^2 - c'^2)} . \sqrt{(\mu'^2 - b^2)(c'^2 - \mu'^2)} . \sqrt{(b^2 - \nu'^2) c'^2 - \nu'^2} .
\end{aligned}
$$

From equations (24) . . . (28), we see that (19) now takes the form

$$
\lambda . \left[\begin{array}{l}
\quad 2A_1 \{ \beta'_1 (\beta'_1 + c'^2)(Y^2 - X^2) + \beta'_1 (\beta'_1 + b^2)(Z^2 - X^2) \} \\
+ 2A_2 \{ \beta'_2 (\beta'_2 + c'^2)(Y^2 - X^2) + \beta'_2 (\beta'_2 + b^2)(Z^2 - X^2) \} \\
+ B_1 \dfrac{2\rho^2 - b^2}{\rho \sqrt{\rho^2 - b^2}} XY + B_2 \dfrac{2\rho^2 - c'^2}{\rho \sqrt{\rho^2 - c'^2}} XZ + B_3 \dfrac{2\rho^2 - b^2 - c'^2}{\sqrt{(\rho^2 - b^2)(\rho^2 - c'^2)}} YZ
\end{array} \right]
$$

$$
- \frac{2\omega i}{\rho \sqrt{(\rho^2 - b^2)}} \left[\begin{array}{l}
- 2A_1 b^2 (\beta'_1 + c'^2)(\rho^2 + \beta'_1) XY - 2A_2 b^2 (\beta'_2 + c'^2)(\rho^2 + \beta'_2) XY \\
+ B_1 \rho \sqrt{\rho^2 - b^2} (X^2 - Y^2) - B_2 \rho \sqrt{\rho^2 - c'^2} YZ \\
\qquad\qquad\qquad\qquad + B_3 \sqrt{(\rho^2 - b^2)(\rho^2 - c'^2)} XZ
\end{array} \right]
$$

$$
\equiv - \lambda^3 \tau^2 . \left[\theta'_1 \frac{c^2 - b^2}{\sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}} YZ + \theta'_2 \frac{-c^2}{\rho \sqrt{\rho^2 - c^2}} ZX + \theta'_3 \frac{b^2}{\rho \sqrt{\rho^2 - b^2}} XY \right] \quad (29).
$$

§ 6. *Calculation of Coefficients in* ψ_1.

Equating coefficients of $Y^2 - X^2$, $Z^2 - X^2$ in the two members of (29)

$$
\left. \begin{array}{ll}
A_1 \beta'_1 (\beta'_1 + c'^2) + A_2 \beta'_2 (\beta'_2 + c'^2) + \dfrac{\omega i}{\lambda} B_1 & = 0 \\
A_1 \beta'_1 (\beta'_1 + b^2) + A_2 \beta'_2 (\beta'_2 + b^2) & = 0
\end{array} \right\} \quad . \quad . \quad . \quad . \quad (30).
$$

Multiply the first of these by 2 and add to the second. Reducing by means of (23) we obtain

$$
b^2 . [- A_1 (c'^2 + \beta'_1) - A_2 (c'^2 + \beta'_2)] + \frac{2\omega i}{\lambda} B_1 = 0 \quad . \quad . \quad . \quad . \quad (31).
$$

Multiply by $\frac{\rho^2}{b^2}$ and subtract from the first of equations (30) ; then

$$
A (\rho^2 + \beta'_1)(c'^2 + \beta'_1) + A_2 (\rho^2 + \beta'_2)(c'^2 + \beta'_2) + \frac{\omega i}{\lambda} \left(1 - \frac{2\rho^2}{b^2} \right) B_1 = 0.
$$

But if we equate coefficients of XY in (29) we obtain

$$B_1(2\rho^2 - b^2) + \frac{4\omega i}{\lambda} b^2 . [A_1(\rho^2 + \beta'_1)(c'^2 + \beta'_1) + A_2(\rho^2 + \beta'_2)(c'^2 + \beta'_2)] = -\lambda^2\tau^2 b^2\theta'_3.$$

Hence

$$B_1(2\rho^2 - b^2) + \frac{4\omega^2}{\lambda^2} B_1(b^2 - 2\rho^2) = -\lambda^2\tau^2\theta'_3 b^2,$$

or

$$B_1(2\rho^2 - b^2)\tau^2 = -\lambda^2\tau^2\theta'_3 b^2,$$

or

$$B_1 = -\frac{\lambda^2 b^2}{2\rho^2 - b^2}\theta'_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Again, equating coefficients of XZ, YZ in (29), we have

$$B_2 \frac{2\rho^2 - c'^2}{\rho\sqrt{(\rho^2 - c'^2)}} - 2\frac{\omega i}{\lambda} B_3 \frac{\sqrt{(\rho^2 - c'^2)}}{\rho} = \lambda^2\tau^2\theta'_2 \frac{c^2}{\rho\sqrt{(\rho^2 - c^2)}},$$

$$B_3 \frac{2\rho^2 - b^2 - c'^2}{\sqrt{(\rho^2 - b^2)(\rho^2 - c'^2)}} + 2\frac{\omega i}{\lambda} B_2 \frac{\sqrt{(\rho^2 - c'^2)}}{\sqrt{(\rho^2 - b^2)}} = -\lambda^2\tau^2\theta'_1 \frac{c^2 - b^2}{\sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}},$$

or, since $\rho^2 - c^2 = \tau^2(\rho^2 - c'^2)$

$$\left.\begin{aligned} B_2(2\rho^2 - c'^2) - \frac{2\omega i}{\lambda} B_3(\rho^2 - c'^2) &= \lambda^2\tau\theta'_2 c^2 \\ B_3(2\rho^2 - b^2 - c'^2) + \frac{2\omega i}{\lambda} B_2(\rho^2 - c'^2) &= -\lambda^2\tau\theta'_1(c^2 - b^2) \end{aligned}\right\} \quad . \quad . \quad . \quad (33).$$

Now, as we have seen above,

$$\begin{aligned} &A_1(\rho'^2 + \beta'_1)(\mu'^2 + \beta'_1)(\nu'^2 + \beta'_1) + A_2(\rho'^2 + \beta'_2)(\mu'^2 + \beta'_2)(\nu'^2 + \beta'_2) \\ &\quad = A_1\{\beta'_1(\beta'_1 + b^2)(\beta'_1 + c'^2) + x^2(\beta'_1 + b^2)(\beta'_1 + c'^2) + y^2\beta'_1(\beta'_1 + c'^2) + z'^2\beta'_1(\beta'_1 + b^2)\} \\ &\quad + A_2\{\beta'_2(\beta'_2 + b^2)(\beta'_2 + c'^2) + x^2(\beta'_2 + b^2)(\beta'_2 + c'^2) + y^2\beta'_2(\beta'_2 + c'^2) + z'^2\beta'_2(\beta'_2 + b^2)\} \end{aligned}$$

which by (30) is equal to

$$\begin{aligned} &A_1\beta'^2_1 . (\beta'_1 + b^2) + A_2\beta'^2_2(\beta'_2 + b^2) + x^2c'^2\{A_1(\beta'_1 + b^2) + A_2(\beta'_2 + b^2)\} + y^2\left\{-\frac{\omega i}{\lambda} B_1\right\} \\ &= A_1(\beta'_1 + b^2)\{-\tfrac{2}{3}(b^2 + c'^2)\beta'_1 - \tfrac{1}{3}b^2c'^2\} + A_2(\beta'_2 + b^2)\{-\tfrac{2}{3}(b^2 + c'^2)\beta'_2 - \tfrac{1}{3}b^2c'^2\} \\ &\qquad + \frac{\omega i}{\lambda} B_1(x^2 - y^2), \end{aligned}$$

and this by means of (30) and (31) reduces to

$$\{-\tfrac{1}{3}b^2 + (x^2 - y^2)\}\frac{\omega i}{\lambda}B_1.$$

Hence the complete value of ψ_1 is

$$\frac{\omega i}{\lambda}B_1\{-\tfrac{1}{3}b^2 + x^2 - y^2\} + B_1xy + B_2xz' + B_3yz'$$

$$= \frac{\omega i}{\lambda}B_1\{-\tfrac{1}{3}b^2 + x^2 - y^2\} + B_1xy + \frac{B_2}{\tau}xz + \frac{B_3}{\tau}yz \quad . \quad . \quad . \quad . \quad (34)$$

where B_1, B_2, B_3 are given in terms of θ'_1, θ'_2, θ'_3 by equations (32), (33).

From equations (6), (34) it will be seen that, in the motions with which we are dealing, u, v, w are linear functions of x, y, z. Hence the components of molecular rotation of the fluid, which involve first differential coefficients of u, v, w, will be independent of x, y, z. This justifies the assumption made in § 1 of the paper.

§ 7. *Calculation of Couples on the Shell due to Fluid Pressure.*

At any point of the fluid the pressure is given by (1); we have, viz.:—

$$p = \rho_1\{V_1 + \tfrac{1}{2}\omega^2(x^2 + y^2)\} - \rho_1\psi.$$

Let us now refer to a new set of rectangular axes, Ox_1, Oy_1, Oz_1, coincident with the principal axes of the ellipsoid in its displaced position. The direction-cosines of one set of axes relatively to the other are given by the scheme

	x_1	y_1	z_1
x	1	$-\theta_3$	θ_2
y	θ_3	1	$-\theta_1$
z	$-\theta_2$	θ_1	1

. . . . (35).

Hence

$$x = x_1 - y_1\theta_3 + z_1\theta_2,$$
$$y = y_1 - z_1\theta_1 + x_1\theta_3,$$
$$z = z_1 - x_1\theta_2 + y_1\theta_1,$$

and, neglecting squares of θ_1, θ_2, θ_3, we obtain

$$p = \rho_1 [V_1 + \tfrac{1}{2}\omega^2 (x_1^2 + y_1^2) - \omega^2 y_1 z_1 \theta_1 + \omega^2 x_1 z_1 \theta_2] - \rho_1 \psi \quad . \quad . \quad . \quad (36),$$

where in the small term ψ we may replace x, y, z by x_1, y_1, z_1.

If L, M, N denote the couples on the shell about the axes Ox_1, Oy_1, Oz_1

$$L = \iint p \, d\sigma \left\{ \frac{Py_1 z_1}{\rho^2 - c^2} - \frac{Py_1 z_1}{\rho^2 - b^2} \right\} = \frac{c^2 - b^2}{(\rho^2 - b^2)(\rho^2 - c^2)} \iint Ppy_1 z_1 \, d\sigma,$$

$$M = \iint p \, d\sigma \left\{ \frac{Px_1 z_1}{\rho^2} - \frac{Px_1 z_1}{\rho^2 - c^2} \right\} = \frac{-c^2}{\rho^2 (\rho^2 - c^2)} \iint Ppz_1 x_1 \, d\sigma,$$

$$N = \iint p \, d\sigma \left\{ \frac{Px_1 y_1}{\rho^2 - b^2} - \frac{Px_1 y_1}{\rho^2} \right\} = \frac{b^2}{\rho^2 (\rho^2 - b^2)} \iint Ppx_1 y_1 \, d\sigma,$$

where $d\sigma$ is an element of the surface of the displaced ellipsoid, and the integrals are taken over the whole surface.

Let us now consider separately the parts of these couples introduced by the different terms in the expression (36) for p.

(a) Take $p = \rho_1 V_1$.

The pressure at every point is the same as if the fluid were at rest under a potential V_1.

V_1 will, in general, consist of three parts due respectively to (α) the attraction of the shell; (β) the mutual attraction of the fluid particles; (γ) any external attracting system.

If the part (α) gave rise to any couple, it would be exactly counterbalanced by the couple on the shell due to the attraction of the fluid, since the attractions of the shell on the fluid and of the fluid on the shell are equal and opposite.

The system of forces (β) also form a system in equilibrium, and, therefore, can give rise to no resultant couple on the shell. Thus no couple can arise from the mutual attractions of the parts of the system.

The pressure at any point due to the part (γ) is the same as if the fluid were at rest. Thus the couples due to any external attracting system will be the same as if the fluid were supposed to be solidified. If we add to these couples, due to the attraction of the external system on the fluid, the parts due to the direct attraction on the shell, we see that the total couples due to any external system will be the same as if our system were solid throughout.

(b.) Take

$$p = \tfrac{1}{2}\omega^2 \rho_1 (x_1^2 + y_1^2) - \omega^2 \rho_1 z_1 (y_1 \theta_1 - x_1 \theta_2).$$

Integrating over the surface of the ellipsoid

$$\frac{x_1^2}{\rho^2} + \frac{y_1^2}{\rho^2 - b^2} + \frac{z_1^2}{\rho^2 - c^2} = 1,$$

we have

$$\iint \mathrm{P}\, d\sigma\, x_1^2 y_1 z_1 = 0, \quad \iint \mathrm{P}\, d\sigma\, y_1^3 z_1 = 0,$$

$$\iint \mathrm{P}\, d\sigma\, y_1^2 z_1^2 = \tfrac{4}{15}\pi\rho\,(\rho^2 \quad b^2)^{\frac{3}{2}}\,(\rho^2 - c^2)^{\frac{3}{2}}, \text{ \&c.}$$

Therefore,

$$\iint p\mathrm{P}\, d\sigma\, y_1 z_1 = -\,\omega^2\theta_1\rho_1 \,.\, \tfrac{4}{15}\pi\rho\,(\rho^2 - b^2)^{\frac{3}{2}}\,(\rho^2 - c^2)^{\frac{3}{2}},$$

and the corresponding part of L is

$$-\,\tfrac{4}{15}\pi\rho_1 \,.\, \rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,(c^2 - b^2)\,\omega^2\theta_1.$$

Similarly the parts of M, N arising from this part of p are

$$+\,\tfrac{4}{15}\pi\rho_1 \,.\, \rho\,(\rho^2 - b^2)^{\frac{1}{2}} \,.\, (\rho^2 - c^2)^{\frac{1}{2}}\,(-\,c^2)\,\omega^2\theta_2 \text{ and } 0.$$

(c.) Lastly, if

$$p = -\,\rho_1\psi = -\,\rho_1 e^{i\lambda t}\left[\frac{\omega i}{\lambda}\mathrm{B}_1\left\{x_1^2 - y_1^2 - \tfrac{1}{3}b^2\right\} + \mathrm{B}_1 x_1 y_1 + \frac{\mathrm{B}_2}{\tau}x_1 z_1 + \frac{\mathrm{B}_3}{\tau}y_1 z_1\right],$$

$$\frac{c^2 - b^2}{(\rho^2 - b^2)\,(\rho^2 - c^2)}\iint \mathrm{P}p\, y_1 z_1\, d\sigma = -\,\tfrac{4}{15}\pi\rho_1\,(c^2 - b^2)\,\rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,\frac{\mathrm{B}_3}{\tau}\,e^{i\lambda t},$$

$$-\,\frac{c^2}{\rho^2\,(\rho^2 - c^2)}\iint \mathrm{P}p\, z_1 x_1\, d\sigma = +\,\tfrac{4}{15}\pi\rho_1 c^2\rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,\frac{\mathrm{B}_2}{\tau}\,e^{i\lambda t},$$

$$\frac{b^2}{\rho^2\,(\rho^2 - b^2)}\iint \mathrm{P}p\, y_1 x_1\, d\sigma = -\,\tfrac{4}{15}\pi\rho_1 b^2\rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,\mathrm{B}_1 e^{i\lambda t}.$$

Collecting the different parts, we obtain for the couples, provided there be no external disturbing force,

$$\left.\begin{aligned}
\mathrm{L} &= -\,\tfrac{4}{15}\pi\rho_1 \,.\, \rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,(c^2 - b^2) \,.\, \left\{\omega^2\theta_1 + \frac{\mathrm{B}_3}{\tau}e^{i\lambda t}\right\} \\
&= -\,\nu \,.\, \left\{\omega^2\theta'_1 + \frac{\mathrm{B}_3}{\tau}\right\}e^{i\lambda t} \\
\mathrm{M} &= +\,\tfrac{4}{15}\pi\rho_1 \,.\, \rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,c^2 \,.\, \left\{\frac{\mathrm{B}_2}{\tau}e^{i\lambda t} - \omega^2\theta_2\right\} \\
&= \mu\left\{\frac{\mathrm{B}_2}{\tau} - \omega^2\theta'_2\right\}e^{i\lambda t} \\
\mathrm{N} &= -\,\tfrac{4}{15}\pi\rho_1 \,.\, \rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,b^2 \,.\, \mathrm{B}_1 e^{i\lambda t} \\
&= (\mu - \nu)\,\mathrm{B}_1 e^{i\lambda t}
\end{aligned}\right\} \quad . \quad (37),$$

where, for brevity, we have put

$$\nu = \tfrac{4}{15}\pi\rho_1 . \rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,(c^2 - b^2), \quad \mu = \tfrac{4}{15}\pi\rho_1 . \rho\,(\rho^2 - b^2)^{\frac{1}{2}}\,(\rho^2 - c^2)^{\frac{1}{2}}\,c^2 \quad . \quad (38).$$

The terms in ω^2 arising from (b) are due to the centrifugal force of the fluid, and occur in consequence of the axis of rotation of the shell not accurately coinciding with that of the fluid, while the terms (c) are due to the effective inertia of the fluid.

§ 8. *Dynamical Equations of Motion of the Shell.*

Let A, B, C be the principal moments of inertia of the shell; p, q, r the angular velocities about the principal axes.

The position of the shell at time $t + \delta t$ may be found from its position at time t by giving it a small rotation $\omega\,\delta t$ about Oz, followed by rotations $\frac{d\theta_1}{dt}\,\delta t$, $\frac{d\theta_2}{dt}\,\delta t$, $\frac{d\theta_3}{dt}\,\delta t$ about the positions of the axes Ox, Oy, Oz at time $t + \delta t$.

The direction-cosines of these latter axes referred to their position at time t are

$$(1,\ \omega\,\delta t,\ 0), \qquad (\omega\,\delta t,\ 1,\ 0), \qquad (0\ ,\ 0\ ,\ 1).$$

Hence, resolving the rotations in the directions of the axes Ox, Oy, Oz at time t, the component angular displacements are

$$\dot{\theta}_1\,\delta t, \qquad \dot{\theta}_2\,\delta t, \qquad (\omega + \dot{\theta}_3)\,\delta t,$$

and the angular velocities about the axes Ox, Oy, Oz are

$$\dot{\theta}_1, \qquad \dot{\theta}_2, \qquad \omega + \dot{\theta}_3.$$

Resolving these about the axes Ox_1, Oy_1, Oz_1 we see from the scheme (35) that

$$p = \dot{\theta}_1 - \omega\theta_2, \qquad q = \dot{\theta}_2 + \omega\theta_1, \qquad r = \omega + \dot{\theta}_3 \quad . \quad . \quad . \quad . \quad (39).$$

EULER's equations of motion are

$$\mathrm{A}\dot{p} - (\mathrm{B} - \mathrm{C})\,qr = \mathrm{L},$$

$$\mathrm{B}\dot{q} - (\mathrm{C} - \mathrm{A})\,rp = \mathrm{M},$$

$$\mathrm{C}\dot{r} - (\mathrm{A} - \mathrm{B})\,pq = \mathrm{N},$$

where, if the system be subject to no external disturbing force, L, M, N have the

values given in (37). If there be any external attracting system we must add to the right-hand members the couples due to this system, estimated as though the system were rigid throughout. In dealing with the "free" oscillations, however, we may omit these terms.

Introducing the values of p, q, r from (39) and omitting small quantities of the second order

$$\begin{aligned} A\ddot{\theta}_1 - (B - C)\,\omega^2\theta_1 - \omega\dot{\theta}_2\,(A + B - C) &= L, \\ B\ddot{\theta}_2 + (C - A)\,\omega^2\theta_2 + \omega\dot{\theta}_1\,(A + B - C) &= M, \\ C\ddot{\theta}_3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad &= N. \end{aligned}$$

Hence, on replacing L, M, N by their values (37), putting $\theta_1 = \theta'_1 e^{i\lambda t}$, &c., and omitting the time factor, we obtain

$$\left.\begin{aligned} \theta'_1\{A\lambda^2 + (B - C)\,\omega^2\} + i\lambda\omega\theta'_2\,(A + B - C) - \nu\left\{\omega^2\theta'_1 + \frac{B_3}{\tau}\right\} &= 0 \\ \theta'_2\{B\lambda^2 + (A - C)\,\omega^2\} - i\lambda\omega\theta'_1\,(A + B - C) - \mu\left\{\omega^2\theta'_2 - \frac{B_2}{\tau}\right\} &= 0 \end{aligned}\right\} \quad . \quad (40)$$

$$\theta'_3\,\{C\lambda^2\} \qquad - (\mu - \nu)\,B_1 \qquad = 0$$

from (32)

$$B_1 = -\frac{\lambda^2 b^2}{2\rho^2 - b^2}\,\theta'_3.$$

Hence the freedom defined by the coordinate θ_3 is neutral, and a disturbance, which causes the coordinate θ_3 to vary, will not give rise to an oscillatory motion.

From (33) the values of B_2/τ, B_3/τ are given by

$$\left.\begin{aligned} \frac{B_2}{\tau}(2\rho^2 - c'^2) - \frac{2\omega i}{\lambda}\frac{B_3}{\tau}(\rho^2 - c'^2) - \lambda^2 c^2\,\theta'_2 &= 0 \\ \frac{B_3}{\tau}(2\rho^2 - b^2 - c'^2) + \frac{2\omega i}{\lambda}\frac{B_2}{\tau}(\rho^2 - c'^2) + \lambda^2\,(c^2 - b^2)\,\theta'_1 &= 0 \end{aligned}\right\} \quad . \quad (41).$$

Eliminating θ'_1, θ'_2, B_2/τ, B_3/τ from (40), (41) by means of a determinant, the periods of free oscillation are given by

$$\begin{vmatrix} A\lambda^2 + (B - C - \nu)\,\omega^2, & i\lambda\omega\,(A + B - C), & 0, & -\nu \\ -i\lambda\omega\,(A + B - C), & B\lambda^2 + (A - C - \mu)\,\omega^2, & \mu, & 0 \\ 0, & -\lambda^2 c^2, & 2\rho^2 - c'^2, & -\frac{2\omega i}{\lambda}(\rho^2 - c'^2) \\ \lambda^2\,(c^2 - b^2), & 0, & \frac{2\omega i}{\lambda}(\rho^2 - c'^2), & 2\rho^2 - b^2 - c'^2 \end{vmatrix} = 0.$$

§ 9. *Reduction of Period Equation.*

Expanding out this determinant, we obtain

$$
\begin{aligned}
&[\{A\lambda^2 - (C - B + \nu)\,\omega^2\}\,\{B\lambda^2 - (C - A + \mu)\,\omega^2\} - \omega^2\lambda^2\,(A + B - C)^2] \\
&\qquad\qquad\qquad \times \left[(2\rho^2 - c'^2)\,(2\rho^2 - b^2 - c'^2) - \frac{4\omega^2}{\lambda^2}\,(\rho^2 - c'^2)^2\right] \\
&\qquad + \mu c^2\lambda^2\,\{A\lambda^2 - (C - B + \nu)\,\omega^2\}\,(2\rho^2 - b^2 - c'^2) \\
&\qquad + \nu\,(c^2 - b^2)\,\lambda^2\,\{B\lambda^2 - (C - A + \mu)\,\omega^2\}\,(2\rho^2 - c'^2) \\
&\qquad - 2\omega^2\lambda^2\,(\rho^2 - c'^2)\,(A + B - C)\,\{\mu\,(c^2 - b^2) + \nu c^2\} \\
&\qquad + \mu\nu c^2\,(c^2 - b^2)\,\lambda^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).
\end{aligned}
$$

Now since

$$
\rho^2 - c^2 = \tau^2\,(\rho^2 - c'^2) = \left(1 - \frac{4\omega^2}{\lambda^2}\right)(\rho^2 - c'^2),
$$

$$
\left.
\begin{aligned}
&(2\rho^2 - c'^2)\,(2\rho^2 - b^2 - c'^2) - \frac{4\omega^2}{\lambda^2}\,(\rho^2 - c'^2)^2 \\
&\qquad = \rho^2\,(\rho^2 - b^2) + (\rho^2 + \overline{\rho^2 - b^2})\,(\rho^2 - c'^2) + (\rho^2 - c'^2)^2\left(1 - \frac{4\omega^2}{\lambda^2}\right) \\
&\qquad = \frac{1}{\tau^2}\cdot\{\tau^2\rho^2\,(\rho^2 - b^2) + (2\rho^2 - b^2)\,(\rho^2 - c^2) + (\rho^2 - c^2)^2\} \\
&\qquad = \frac{1}{\lambda^2\tau^2}\cdot\{\lambda^2\,(2\rho^2 - c^2)\,(2\rho^2 - b^2 - c^2) - 4\omega^2\rho^2\,(\rho^2 - b^2)\} \\
&2\rho^2 - b^2 - c'^2 = \frac{1}{\tau^2}\cdot\{\tau^2\,(\rho^2 - b^2) + (\rho^2 - c^2)\} \\
&\qquad\qquad = \frac{1}{\lambda^2\tau^2}\cdot\{\lambda^2\,(2\rho^2 - b^2 - c^2) - 4\omega^2\,(\rho^2 - b^2)\} \\
&\text{and} \\
&2\rho^2 - c'^2 = \frac{1}{\tau^2}\cdot\{\tau^2\rho^2 + (\rho^2 - c^2)\} = \frac{1}{\lambda^2\tau^2}\cdot\{\lambda^2\,(2\rho^2 - c^2) - 4\omega^2\rho^2\}
\end{aligned}
\right\} \quad . \quad (43).
$$

Hence substituting in (42) we obtain the following cubic for λ^2:

$$
\begin{aligned}
&[\{A\lambda^2 - (C - B + \nu)\,\omega^2\}\,\{B\lambda^2 - (C - A + \mu)\,\omega^2\} - \omega^2\lambda^2\,(A + B - C)^2] \\
&\qquad\qquad\qquad \times \{\lambda^2\,(2\rho^2 - b^2 - c^2)\,(2\rho^2 - c^2) - 4\omega^2\rho^2\,(\rho^2 - b^2)\} \\
&\quad + \mu c^2\lambda^2.\{A\lambda^2 - (C - B + \nu)\,\omega^2\}\,\{\lambda^2\,(2\rho^2 - b^2 - c^2) - 4\omega^2\,(\rho^2 - b^2)\} \\
&\quad + \nu\,(c^2 - b^2)\,\lambda^2.\{B\lambda^2 - (C - A + \mu)\,\omega^2\}\,\{\lambda^2\,(2\rho^2 - c^2) - 4\omega^2\rho^2\} \\
&\quad - 2\omega^2\lambda^4.(\rho^2 - c^2)\,(A + B - C)\,\{\mu\,(c^2 - b^2) + \nu c^2\} \\
&\quad + \mu\nu\lambda^4.(\lambda^2 - 4\omega^2)\,c^2.(c^2 - b^2) = 0.
\end{aligned}
$$

Let us now change our notation, replacing ρ^2, $\rho^2 - b^2$, $\rho^2 - c^2$ by α^2, β^2, γ^2; the period equation may then be written

$$\begin{aligned}
&[\{A\lambda^2 + (B - C - \nu)\,\omega^2\}\,\{B\lambda^2 + (A - C - \mu)\,\omega^2\} - \omega^2\lambda^2(A + B - C)^2] \\
&\qquad\qquad \times [\lambda^2(\alpha^2 + \gamma^2)(\beta^2 + \gamma^2) - 4\omega^2\alpha^2\beta^2] \\
&+ \mu(\alpha^2 - \gamma^2)\,\lambda^2\,\{A\lambda^2 + (B - C - \nu)\,\omega^2\}\,\{\lambda^2(\beta^2 + \gamma^2) - 4\omega^2\beta^2\} \\
&+ \nu(\beta^2 - \gamma^2)\,\lambda^2\,\{B\lambda^2 + (A - C - \mu)\,\omega^2\}\,\{\lambda^2(\alpha^2 + \gamma^2) - 4\omega^2\alpha^2\} \\
&- 2\omega^2\lambda^4\gamma^2\,.\,(A + B - C)\,\{\mu(\beta^2 - \gamma^2) + \nu(\alpha^2 - \gamma^2)\} \\
&+ \mu\nu\lambda^4(\lambda^2 - 4\omega^2)(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).
\end{aligned}$$

If we put $\lambda^2 = \omega^2$, the left-hand member of (44) becomes

$$\begin{aligned}
&\omega^6[(A + B - C - \nu)(A + B - C - \mu) - (A + B - C)^2][(\alpha^2 + \gamma^2)(\beta^2 + \gamma^2) - 4\alpha^2\beta^2] \\
&\quad + \omega^6\mu(\alpha^2 - \gamma^2)\,\{A + B - C - \nu\}\,\{\gamma^2 - 3\beta^2\} \\
&\quad + \omega^6\nu(\beta^2 - \gamma^2)\,\{A + B - C - \mu\}\,\{\gamma^2 - 3\alpha^2\} \\
&\quad - 2\omega^6\gamma^2(A + B - C)\,\{\mu(\beta^2 - \gamma^2) + \nu(\alpha^2 - \gamma^2)\} - 3\mu\nu\omega^6(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2) \\
&= -\omega^6(A + B - C)\left[\begin{array}{l}\mu\,\{(\alpha^2+\gamma^2)(\beta^2+\gamma^2) - 4\alpha^2\beta^2 + (\gamma^2-\alpha^2)(\gamma^2-3\beta^2) - 2\gamma^2(\gamma^2-\beta^2)\} \\ +\nu\{(\alpha^2+\gamma^2)(\beta^2+\gamma^2) - 4\alpha^2\beta^2 + (\gamma^2-\beta^2)(\gamma^2-3\alpha^2) - 2\gamma^2(\gamma^2-\alpha^2)\}\end{array}\right] \\
&\quad + \mu\nu\omega^6\left[\begin{array}{l}(\alpha^2 + \gamma^2)(\beta^2 + \gamma^2) - 4\alpha^2\beta^2 + (\gamma^2 - \alpha^2)(\gamma^2 - 3\beta^2) \\ \qquad + (\gamma^2 - \beta^2)(\gamma^2 - 3\alpha^2) - 3(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)\end{array}\right] = 0\,;
\end{aligned}$$

therefore $\lambda^2 = \omega^2$ is always one root of (44).

Arranging (44) according to powers of λ^2, we have

$$\begin{aligned}
&\lambda^6\left[\begin{array}{l}AB(\alpha^2 + \gamma^2)(\beta^2 + \gamma^2) + \mu A(\beta^2 + \gamma^2)(\alpha^2 - \gamma^2) + \nu B(\alpha^2 + \gamma^2)(\beta^2 - \gamma^2) \\ + \mu\nu(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)\end{array}\right] \\
&-\omega^2\lambda^4.\left[\begin{array}{l}(\alpha^2+\gamma^2)(\beta^2+\gamma^2)\,\{(A+B-C)^2 - A(A-C-\mu) - B(B-C-\nu)\} + 4AB\alpha^2\beta^2 \\ - \mu(\alpha^2 - \gamma^2)\,\{(\beta^2 + \gamma^2)(B - C - \nu) - 4A\beta^2\} \\ - \nu(\beta^2 - \gamma^2)\,\{(\alpha^2 + \gamma^2)(A - C - \mu) - 4B\alpha^2\} \\ + 2\gamma^2(A + B - C)\{\mu(\beta^2 - \gamma^2) + \nu(\alpha^2 - \gamma^2)\} \\ + 4\mu\nu(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)\end{array}\right] \\
&\quad + \omega^4\lambda^2.\left[\begin{array}{l}(\alpha^2 + \gamma^2)(\beta^2 + \gamma^2)(B - C - \nu)(A - C - \mu) \\ + 4\alpha^2\beta^2\,\{(A + B - C)^2 - A(A - C - \mu) - B(B - C - \nu)\} \\ - 4\mu\beta^2(\alpha^2 - \gamma^2)(B - C - \nu) - 4\nu\alpha^2(\beta^2 - \gamma^2)(A - C - \mu)\end{array}\right] \\
&\qquad\qquad - \omega^6\,\{4\alpha^2\beta^2(B - C - \nu)(A - C - \mu)\}\,. \quad . \quad . \quad . \quad . \quad . \quad = 0.
\end{aligned}$$

Dividing out by the factor $\lambda^2 - \omega^2$, we are left with

$$\lambda^4 . [A(\beta^2+\gamma^2)+\nu(\beta^2-\gamma^2)][B(\alpha^2+\gamma^2)+\mu(\alpha^2-\gamma^2)]$$
$$-\omega^2\lambda^2\left[\begin{array}{l}(\alpha^2+\gamma^2)(\beta^2+\gamma^2)(B-C-\nu)(A-C-\mu)\\ +4\alpha^2\beta^2\{(A+B-C)^2-A(A-C-\mu)-B(B-C-\nu)-(B-C-\nu)(A-C-\mu)\}\\ -4\mu\beta^2.(\alpha^2-\gamma^2)(B-C-\nu)-4\nu\alpha^2(\beta^2-\gamma^2)(A-C-\mu)\end{array}\right]$$
$$+\omega^4 . \{4\alpha^2\beta^2(B-C-\nu)(A-C-\mu)\}=0,$$

or

$$\lambda^4 . [A(\beta^2+\gamma^2)+\nu(\beta^2-\gamma^2)][B(\alpha^2+\gamma^2)+\mu(\alpha^2-\gamma^2)]$$
$$-\omega^2\lambda^2\left[\begin{array}{l}(\alpha^2+\gamma^2)(\beta^2+\gamma^2)(B-C-\nu)(A-C-\mu)+4\alpha^2\beta^2(A+\nu)(B+\mu)\\ +4\mu\beta^2\gamma^2.(B-C-\nu)+4\nu\alpha^2\gamma^2(A-C-\mu)\end{array}\right]$$
$$+\omega^4\{4\alpha^2\beta^2(B-C-\nu)(A-C-\mu)\}=0 \quad . \quad . \quad . \quad . \quad . \quad (45).$$

In order that the system may satisfy the criteria for "ordinary" stability the roots of this equation in λ^2 must be real and positive.

The period equation (45) agrees with the equation (5) (§ 1), and the solution of it, in the case in which the ellipsoid differs but slightly from a sphere is given in § 3.

§ 10. *Nature of the Oscillations.*

From equations (40), (41), (43) we see that the equations giving the ratios of the quantities θ'_1, θ'_2, B_2/τ, B_3/τ are

$$\left.\begin{array}{l}\theta'_1\{A\lambda^2+(B-C)\omega^2\}+i\lambda\omega\,\theta'_2(A+B-C)-\nu\left(\omega^2\theta'_1+\frac{B_3}{\tau}\right)=0\\ \theta'_2\{B\lambda^2+(A-C)\omega^2\}-i\lambda\omega\,\theta'_1(A+B-C)-\mu\left(\omega^2\theta'_2+\frac{B_2}{\tau}\right)=0\\ \frac{B_2}{\tau}\{(\alpha^2+\gamma^2)\lambda^2-4\omega^2\alpha^2\}-2\omega i\lambda\gamma^2\frac{B_3}{\tau}-\lambda^2(\lambda^2-4\omega^2)(\alpha^2-\gamma^2)\theta'_2=0\\ \frac{B_3}{\tau}\{(\beta^2+\gamma^2)\lambda^2-4\omega^2\beta^2\}+2\omega i\lambda\gamma^2\frac{B_2}{\tau}+\lambda^2(\lambda^2-4\omega^2)(\beta^2-\gamma^2)\theta'_1=0\end{array}\right\} \quad . \quad (46).$$

(a). We have seen that in every case $\lambda^2=\omega^2$ is one root of the period equation; when $\lambda=\omega$ the equations (46) reduce to

$$\theta'_1\{A+B-C-\nu\}+i\theta'_2(A+B-C)-\nu\frac{B_3}{\tau\omega^2}=0,$$
$$\theta'_2\{A+B-C-\mu\}-i\theta'_1(A+B-C)+\mu\frac{B_2}{\tau\omega^2}=0,$$
$$\frac{B_2}{\tau\omega^2}(\gamma^2-3\alpha^2)-2i\gamma^2\frac{B_3}{\tau\omega^2}+3(\alpha^2-\gamma^2)\theta'_2=0,$$
$$\frac{B_3}{\tau\omega^2}(\gamma^2-3\beta^2)+2i\gamma^2\frac{B_2}{\tau\omega^2}-3(\beta^2-\gamma^2)\theta'_1=0,$$

which are satisfied by

$$\theta'_1 = -i\theta'_2 = -\frac{B_3}{\tau\omega^2} = -i\frac{B_2}{\tau\omega^2}.$$

Suppose $\theta_1 = \phi e^{i(\omega t+\epsilon)}$ where ϕ is a real quantity; we have as one solution

$$\theta_1 = \phi e^{i(\omega t+\epsilon)}, \qquad \theta_2 = i\phi e^{i(\omega t+\epsilon)}.$$

Changing the sign of i wherever it occurs, another solution is given by

$$\theta_1 = \phi e^{-i(\omega t+\epsilon)}, \qquad \theta_2 = -i\phi e^{-i(\omega t+\epsilon)}$$

and this corresponds to the value $-\omega$ of λ.

Combining these two solutions we get as the real motion corresponding to the root $\lambda^2 = \omega^2$,

$$\theta_1 = 2\phi\cos(\omega t+\epsilon), \qquad \theta_2 = -2\phi\sin(\omega t+\epsilon).$$

Now θ_1, θ_2 serve to determine the position of that principal axis which in the steady motion coincides with the axis of rotation, relatively to axes which are themselves revolving with angular velocity ω.

Let us consider the angular displacements relatively to fixed axes $O\xi$, $O\eta$, $O\zeta$ coincident with the position occupied by the moving axes at the time $t = 0$; they are

$$\theta_1\cos\omega t - \theta_2\sin\omega t = 2\phi\cos\epsilon,$$
$$\theta_1\sin\omega t + \theta_2\cos\omega t = -2\phi\sin\epsilon,$$

and these are constant quantities. Thus, the apparent oscillation which corresponds to the root $\lambda^2 = \omega^2$, consists of a small permanent displacement of the axis of rotation, and the system rotates as if rigid about an axis which does not accurately coincide with our axis Oz. It is obvious that if ω be the angular velocity of rotation about this axis, the system and the moving axes Ox, Oy, Oz will return to their original positions after an interval $2\pi/\omega$, and, therefore, the system will appear to oscillate in a period $2\pi/\omega$ relatively to these moving axes.

It is easy to see that the fluid motion, indicated by the analysis, also consists of a motion of pure rotation.

For when $\theta_1 = \phi e^{i(\omega t+\epsilon)}$

$$B_1 = 0 \qquad B_2/\tau\omega^2 = -iB_3/\tau\omega^2 = i\phi e^{i\epsilon}.$$

Therefore, from (34),

$$\psi_1 = \omega^2 i\phi e^{i\epsilon}\{xz + iyz\},$$

and from (6),

$$u_1 = \frac{1}{3\omega}\{-\phi\omega^2 ze^{i\epsilon} - 2\phi\omega^2 ze^{i\epsilon}\} = -\phi\omega ze^{i\epsilon}$$

$$v_1 = \frac{1}{3\omega}\{-i\phi\omega^2 ze^{i\epsilon} - 2i\phi\omega^2 ze^{i\epsilon}\} = -\phi\omega zie^{i\epsilon}$$

$$w_1 = \phi\omega(x+iy)e^{i\epsilon},$$

whence

$$u = -\phi\omega ze^{i(\omega t+\epsilon)}, \qquad v = -\phi\omega zie^{i(\omega t+\epsilon)}, \qquad w = \phi\omega(x+iy)e^{i(\omega t+\epsilon)},$$

and, in the corresponding real solution,

$$u = -2\phi\omega z\cos(\omega t+\epsilon), \qquad v = 2\phi\omega z\sin(\omega t+\epsilon),$$
$$w = 2\phi\omega[x\cos(\omega t+\epsilon) - y\sin(\omega t+\epsilon)].$$

These are the component velocities relatively to the moving axes. The velocities parallel to the instantaneous positions of the moving axes are

$$u - y\omega, \qquad v + x\omega, \qquad w.$$

The velocities parallel to the fixed $O\xi$, $O\eta$, $O\zeta$, are therefore

$$(u-y\omega)\cos\omega t - (v+x\omega)\sin\omega t = -2\phi\omega z\cos\epsilon - \omega\eta,$$
$$(v+x\omega)\cos\omega t + (u-y\omega)\sin\omega t = 2\phi\omega z\sin\epsilon + \omega\xi,$$
$$w = 2\phi\omega\{\xi\cos\epsilon - \eta\sin\epsilon\}.$$

Thus the fluid motion is a motion of pure rotation, the component angular velocities about the axes being

$$-2\phi\omega\sin\epsilon, \qquad -2\phi\omega\cos\epsilon, \qquad \omega.$$

The resultant of these angular velocities is an angular velocity ω about the line whose direction cosines are

$$-2\phi\sin\epsilon, \qquad -2\phi\cos\epsilon, \qquad 1.$$

The similar case for the spheroid with a free surface has been already discussed by BRYAN ('Phil. Trans.,' 1889).

(*b*). Next take

$$\lambda = \omega[1 + (\epsilon_1+\epsilon_2)(1+q)]^{\frac{1}{2}} = \omega(1+E) \text{ say}$$

where

$$E = \tfrac{1}{2}(\epsilon_1+\epsilon_2)(1+q).$$

Substituting this value of λ in (46) and putting $\alpha = \gamma(1+\epsilon_1)$, &c., we obtain

$$\left.\begin{array}{l}\theta'_1\{A+B-C+2AE-qC\epsilon_2\}+i\theta'_2\{(A+B-C)(1+E)\}-qC\epsilon_2\dfrac{B_3}{\tau\omega^2}=0,\\ \theta'_2\{(A+B-C)+2BE-qC\epsilon_1\}-i\theta'_1\{(A+B-C)(1+E)\}+qC\epsilon_1\dfrac{B_2}{\tau\omega^2}=0,\\ \dfrac{B_2}{\tau\omega^2}\{1+\epsilon_1-E\}+i\dfrac{B_3}{\tau\omega^2}(1+E)-\theta'_2\epsilon_1(3+2E)=0,\\ \dfrac{B_3}{\tau\omega^2}\{1+\epsilon_2-E\}-i\dfrac{B_2}{\tau\omega^2}(1+E)+\theta'_1\epsilon_2(3+2E)=0.\end{array}\right\}\quad(47).$$

These equations are correct as far as first powers of ϵ_1, ϵ_2, E only. Solving for the ratios of θ'_1, θ'_2, $B_2/\tau\omega^2$, $B_3/\tau\omega^2$, we have

$$\frac{\theta'_1}{\begin{vmatrix}A+B-C+2BE-qC\epsilon_1, & qC\epsilon_1, & 0\\ -3\epsilon_1, & 1+\epsilon_1-E, & i(1+E)\\ 0, & -i(1+E), & 1+\epsilon_2-E\end{vmatrix}}=\frac{-\theta'_2}{\begin{vmatrix}-i(A+B-C)(1+E), & qC\epsilon_1, & 0\\ 0 & 1+\epsilon_1-E, & i(1+E)\\ \epsilon_2(3+2E), & -i(1+E), & 1+\epsilon_2-E\end{vmatrix}}$$

$$=\frac{B_2/\tau\omega^2}{\begin{vmatrix}-i(A+B-C)(1+E), & A+B-C+2BE-qC\epsilon_1, & 0\\ 0, & -3\epsilon_1, & i(1+E)\\ 3\epsilon_2, & 0, & 1+\epsilon_2-E\end{vmatrix}}$$

$$=\frac{-B_3/\tau\omega^2}{\begin{vmatrix}-i(A+B-C)(1+E), & A+B-C+2BE-qC\epsilon_1, & qC\epsilon_1\\ 0, & -3\epsilon_1, & 1+\epsilon_1-E\\ 3\epsilon_2 & 0, & -i(1+E)\end{vmatrix}},$$

or

$$\frac{\theta'_1}{(A+B-C)\{\epsilon_1+\epsilon_2-4E\}}=\frac{-i\theta'_2}{(A+B-C)\{\epsilon_1+\epsilon_2-4E\}}=\frac{-iB_2/\tau\omega^2}{3(\epsilon_1+\epsilon_2)(A+B-C)}=\frac{-B_3/\tau\omega^2}{3(\epsilon_1+\epsilon_2)(A+B-C)},$$

where the denominators are correct as far as first powers of ϵ_1, &c., only.

Replacing E by its value $\frac{1}{2}(\epsilon_1+\epsilon_2)(1+q)$ we obtain

$$\frac{\theta'_1}{-(2q+1)}=\frac{i\theta'_2}{2q+1}=\frac{-iB_2/\tau\omega^2}{3}=\frac{-B_3/\tau\omega^2}{3}\quad\ldots\ldots\quad(48).$$

Taking

$$\theta'_1=\phi e^{i\epsilon}$$

we have

$$\theta'_2=i\phi e^{i\epsilon}$$

and, in the corresponding real motion,

$$\theta_1=2\phi\cos(\lambda t+\epsilon),$$
$$\theta_2=-2\phi\sin(\lambda t+\epsilon).$$

The angular displacements, relatively to the fixed axes $O\xi$, $O\eta$, $O\zeta$, at time t are

$$\theta_1 \cos \omega t - \theta_2 \sin \omega t = 2\phi \cos [(\lambda - \omega) t + \epsilon],$$
$$\theta_1 \sin \omega t + \theta_2 \cos \omega t = - 2\phi \sin [(\lambda - \omega) t + \epsilon].$$

Thus the motion of the principal axis consists approximately of a simple harmonic motion in period $2\pi/(\lambda - \omega) = 2\pi/\mathrm{E}\omega$, in virtue of which it describes a small cone about its mean position in a direction opposite to that in which the system is rotating.

The position of the instantaneous axis of rotation of the shell is defined by the direction cosines

$$\dot{\theta}_1/\omega, \qquad \dot{\theta}_2/\omega, \qquad 1,$$

or

$$- 2\phi \frac{\lambda}{\omega} \sin (\lambda t + \epsilon), \qquad - 2\phi \frac{\lambda}{\omega} \cos (\lambda t + \epsilon), \qquad 1\,;$$

and since λ is approximately equal to ω, this axis will be very nearly coincident with the principal axis.

From (48) we have also

$$\frac{\mathrm{B}_2}{\tau\omega^2} = - i \frac{\mathrm{B}_3}{\tau\omega^2} = - \frac{3}{1 + 2q} i\theta'_1 = - \frac{3}{1 + 2q} i\phi e^{i\epsilon},$$

and therefore from (34)

$$\psi_1 = - \frac{3}{1 + 2q} i\phi\omega^2 (xz + iyz)\, e^{i\epsilon}\,;$$

to the same order of approximation we have

$$u_1 = \frac{3}{1 + 2q} \phi\omega z\, e^{i\epsilon}, \quad v_1 = \frac{3}{1 + 2q} \phi\omega z i\, e^{i\epsilon}, \quad \omega_1 = - \frac{3}{1 + 2q} \phi\omega (x + iy)\, e^{i\epsilon}.$$

Whence, in the corresponding real solution

$$u = \frac{6}{1 + 2q} \phi\omega z \cos (\lambda t + \epsilon), \quad v = - \frac{6}{1 + 2q} \phi\omega z \sin (\lambda t + \epsilon),$$
$$w = \frac{6}{1 + 2q} \phi\omega [- x \cos (\lambda t + \epsilon) + y \sin (\lambda t + \epsilon)].$$

The velocity components relative to the fixed axes are therefore

$$(u - y\omega) \cos \omega t - (v + x\omega) \sin \omega t = \frac{6}{1 + 2q} \phi\omega z \cos (\overline{\lambda - \omega} t + \epsilon) - \omega\eta,$$

$$(v + x\omega) \cos \omega t + (u - y\omega) \sin \omega t = - \frac{6}{1 + 2q} \phi\omega z \sin (\overline{\lambda - \omega} t + \epsilon) + \omega\xi,$$

$$w = - \frac{6}{1 + 2q} \phi\omega \{\xi \cos (\overline{\lambda - \omega} t + \epsilon) - \eta \sin (\overline{\lambda - \omega} t + \epsilon)\}.$$

Hence, the motion of the fluid will consist very approximately of a rotation, as if rigid, with angular velocity ω about the line whose direction-cosines are

$$\frac{6}{1+2q}\,\phi\sin(\overline{\lambda-\omega}t+\epsilon),\quad \frac{6}{1+2q}\,\phi\cos(\overline{\lambda-\omega}t+\epsilon),\ 1.$$

This axis will itself describe a cone in period $2\pi/\mathrm{E}\omega$, and it will be so situated that the axis Oz lies in the plane containing the axes of revolution of the fluid and of the shell, and is between these two axes. The semi-vertical angles of the cones described by the axes of rotation of fluid and the shell will be in the ratio $3 : 2q + 1$.

The motion under discussion is that which would ensue if the shell were set rotating about its principal axis, while the fluid possessed a rotatory motion in the same period about some other axis. It is clear, that as ϵ_1, ϵ_2, and consequently E, diminish, the period of this oscillation will be prolonged; that is to say, the motion of the axes of rotation will become slower. This motion will be reduced to zero when ϵ_1, ϵ_2 vanish. In this case the internal surface of the shell is spherical, and the shell and fluid, of course, move independently. So far as this (apparent) oscillation is concerned, they will each be rotating with angular velocity ω, but about different axes.

From the expression for the ratio of the amplitudes, we see that when q is large, that is when the effective inertia of the fluid is large, compared with that of the shell, the disturbance of the shell will be considerable, compared with that of the fluid; whereas if q be small, the disturbance of the shell bears to that of the fluid, a ratio which approximates to, but is always in excess of, $1 : 3$.

This oscillation has been previously examined by HOPKINS ('Phil. Trans.,' 1839) under certain special assumptions, as to the initial circumstances, and to the law of distribution of density in the shell.

(c). Lastly, suppose $\lambda = \omega\sqrt{(\kappa_1+q\epsilon_1)(\kappa_2+q\epsilon_2)}$.

The approximate form of equations (46) is now

$$\theta'_1\{\kappa_2+q\epsilon_2\}\,\mathrm{B} - i\theta'_2\mathrm{B}\sqrt{(\kappa_1+q\epsilon_1)(\kappa_2+q\epsilon_2)} + q\mathrm{C}\epsilon_2\frac{\mathrm{B}_3}{\tau} = 0,$$

$$\theta'_2\{\kappa_1+q\epsilon_1\}\,\mathrm{A} + i\theta'_1\mathrm{A}\sqrt{(\kappa_1+q\epsilon_1)(\kappa_2+q\epsilon_2)} - q\mathrm{C}\epsilon_1\frac{\mathrm{B}_2}{\tau} = 0,$$

$$2\frac{\mathrm{B}_2}{\tau}(1+2\epsilon_1) + i\sqrt{(\kappa_1+q\epsilon_1)(\kappa_2+q\epsilon_2)}\frac{\mathrm{B}_3}{\tau} = 0,$$

$$2\frac{\mathrm{B}_3}{\tau}(1+2\epsilon_2) - i\sqrt{(\kappa_1+q\epsilon_1)(\kappa_2+q\epsilon_2)}\frac{\mathrm{B}_2}{\tau} = 0.$$

Hence approximately $\mathrm{B}_2/\tau = \mathrm{B}_3/\tau = 0$, and

$$\frac{\theta'_1}{\sqrt{(\kappa_1+q\epsilon_1)}} = \frac{i\theta'_2}{\sqrt{(\kappa_2+q\epsilon_2)}} = \phi e^{i\epsilon},\ \text{say};$$

therefore

$$\theta_1 = \phi \sqrt{(\kappa_1 + q\epsilon_1)}\, e^{i(\lambda t + \epsilon)},$$
$$\theta_2 = -\, i\phi \sqrt{(\kappa_2 + q\epsilon_2)}\, e^{i(\lambda t + \epsilon)}.$$

The corresponding real solution is

$$\theta_1 = 2\phi \sqrt{(\kappa_1 + q\epsilon_1)} \cos(\lambda t + \epsilon),$$
$$\theta_2 = 2\phi \sqrt{(\kappa_2 + q\epsilon_2)} \sin(\lambda t + \epsilon).$$

The angular displacements about $O\xi$, $O\eta$ are therefore

$$\theta_1 \cos \omega t - \theta_2 \sin \omega t = \phi \{\sqrt{(\kappa_1 + q\epsilon_1)} + \sqrt{(\kappa_2 + q\epsilon_2)}\} \cos[(\omega + \lambda) t + \epsilon]$$
$$+ \phi \{\sqrt{(\kappa_1 + q\epsilon_1)} - \sqrt{(\kappa_2 + q\epsilon_2)}\} \cos[(\omega - \lambda) t - \epsilon],$$
$$\theta_2 \cos \omega t + \theta_1 \sin \omega t = \phi \{\sqrt{(\kappa_1 + q\epsilon_1)} + \sqrt{(\kappa_2 + q\epsilon_2)}\} \sin[(\omega + \lambda) t + \epsilon]$$
$$+ \phi \{\sqrt{(\kappa_1 + q\epsilon_1)} - \sqrt{(\kappa_2 + q\epsilon_2)}\} \sin[(\omega - \lambda) t - \epsilon]$$

The motion of the principal axis in space consists, therefore, of a combination of two simple harmonic motions, the period of each being approximately equal to the period of rotation of the system, and the amplitudes being in the ratio $\sqrt{(\kappa_1 + q\epsilon_1)} \backsim \sqrt{(\kappa_2 + q\epsilon_2)} : \sqrt{(\kappa_1 + q\epsilon_1)} + \sqrt{(\kappa_2 + q\epsilon_2)}$; in virtue of each of these oscillations, the principal axis will describe a cone of revolution in the direction in which the system is rotating. In the event of the system being symmetrical about the axis of rotation $\kappa_1 = \kappa_2$ and $\epsilon_1 = \epsilon_2$, in this case the amplitude of one of the oscillations reduces to zero.

We have likewise

$$\frac{\dot{\theta}_1}{\omega} = -\, 2\phi \frac{\lambda}{\omega} \sqrt{(\kappa_1 + q\epsilon_1)} \sin(\lambda t + \epsilon),$$

$$\frac{\dot{\theta}_2}{\omega} = +\, 2\phi \frac{\lambda}{\omega} \sqrt{(\kappa_2 + q\epsilon_2)} \cos(\lambda t + \epsilon),$$

$$\frac{\dot{\theta}_1}{\omega} \cos \omega t - \frac{\dot{\theta}_2}{\omega} \sin \omega t = -\, \phi \frac{\lambda}{\omega} \{\sqrt{(\kappa_1 + q\epsilon_1)} + \sqrt{(\kappa_2 + q\epsilon_2)}\} \sin(\overline{\omega + \lambda}\, t + \epsilon)$$
$$-\, \phi \frac{\lambda}{\omega} \{\sqrt{(\kappa_1 + q\epsilon_1)} - \sqrt{(\kappa_2 + q\epsilon_2)}\} \sin(\sqrt{\omega - \lambda}\, t - \epsilon),$$

$$\frac{\dot{\theta}_1}{\omega} \sin \omega t + \frac{\dot{\theta}_2}{\omega} \cos \omega t = \phi \frac{\lambda}{\omega} \{\sqrt{(\kappa_1 + q\epsilon_1)} + \sqrt{(\kappa_2 + q\epsilon_2)}\} \cos\{(\omega + \lambda) t + \epsilon\}$$
$$+ \phi \frac{\lambda}{\omega} \{\sqrt{(\kappa_1 + q\epsilon_1)} - \sqrt{(\kappa_2 + q\epsilon_2)}\} \cos\{(\omega - \lambda) t - \epsilon\}.$$

The motion of the instantaneous axis of rotation of the shell is therefore in all respects similar to that of the principal axis, but the semi-vertical angles of the cones described are smaller in the ratio $\lambda : \omega$.

The direction-cosines of the instantaneous axis referred to the principal axes of the shell Ox_1, Oy_1, Oz_1 are

$$\frac{\dot{\theta}_1 - \omega\theta_2}{\omega}, \quad \frac{\dot{\theta}_2 + \omega\theta_1}{\omega}, \quad 1,$$

or

$$-2\phi\left[\frac{\lambda}{\omega}\sqrt{(\kappa_1 + q\epsilon_1)} + \sqrt{(\kappa_2 + q\epsilon_2)}\right]\sin(\lambda t + \epsilon),$$

$$2\phi\left[\frac{\lambda}{\omega}\sqrt{(\kappa_2 + q\epsilon_2)} + \sqrt{(\kappa_1 + q\epsilon_1)}\right]\cos(\lambda t + \epsilon), \quad 1;$$

therefore, relatively to the shell, the instantaneous axis describes a cone in period $2\pi/\lambda$.

This motion would ensue if the shell were started rotating about an axis not coincident with its principal axis, and it is analogous to the motion of a rigid body under no forces when slightly disturbed from a motion of pure rotation about a principal axis.